Maass relations for Saito-Kurokawa lifts of higher levels

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Satake, 1966: reformulation in terms of automorphic representations, generalisation to modular forms on other groups

Siegel modular forms of degree 2

$$\mathrm{SL}_2(\mathbb{Z}) \circlearrowright_k f : \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\} \to \mathbb{C}$$

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$\rightarrow$ There exists a basis consisting of common eigenforms.

Ramanujan-Petersson conjecture

Definition

For a Siegel modular form F of degree 2, which is an eigenform with eigenvalues $\{\lambda_p, \lambda_{p^2} : p \text{ prime}\}$ the spinor L-function is given by

$$L(s, F) = \prod_p L_p\left(p^{-s}\right)^{-1}, \quad s \in \mathbb{C},$$

where

$$\begin{aligned} L_p(X) &= \text{ polynomial depending on } \lambda_p, \lambda_{p^2} \\ &= (1 - \alpha_{0,p}X)(1 - \alpha_{0,p}\alpha_{1,p}X)(1 - \alpha_{0,p}\alpha_{2,p}X)(1 - \alpha_{0,p}\alpha_{1,p}\alpha_{2,p}X). \end{aligned}$$

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Conjecture (Satake, 1966)

For $j = 1, 2$ and all prime numbers p:

$$|\alpha_{j,p}| = 1.$$

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Let $k \geq 10$ even. Then there exists an injective mapping

$$SK\colon S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})) \to S_k(\mathrm{Sp}_4(\mathbb{Z}))$$

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2. $L(s, SK(f)) = \zeta(s-k+2)\zeta(s-k+1)L(s,f)$,

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $L(s,f) = \sum_{n=1}^{\infty} \frac{a(f,n)}{n^s}$.

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(2 implies that $SK(f)$ does not satisfy Ramanujan-Petersson conjecture.)

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Resnikoff, Saldaña, 1974: discovery of F that satisfy the relation

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$\rightarrow$ *Weissauer, 2009:* Ramanujan-Petersson conjecture holds for F that are not Saito-Kurokawa lifts.

Saito-Kurokawa lifting

$S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})) \xrightarrow{\sim}$ Jacobi forms (weight k, index 1, level 1) $\longrightarrow S_k(\mathrm{Sp}_4(\mathbb{Z}))$

Eichler-Zagier corr. $\circ$ (Shimura lift)$^{-1}$
(through modular forms of weight $k - 1/2$)

due to Maass, 1979

So far we considered only

$$SK\colon S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})) \to S_k(\mathrm{Sp}_4(\mathbb{Z})).$$

What about higher levels?

Representation theoretic prelude

Ring of adeles of $\mathbb{Q}$:

$$\mathbb{A} := \{(a_p)_{p\leq\infty} \in \prod_{p\leq\infty} \mathbb{Q}_p : a_p \in \mathbb{Z}_p \text{ for almost all } p\}.$$

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where

$$\mathrm{GSp}_{2n}(\mathbb{Q}) := \{g \in \mathrm{GL}_{2n}(\mathbb{Q}) : \ {}^t g \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix} g = \mu(g) \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}\}.$$

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Adelisation of modular forms:

$$f \in S_k(\mathrm{SL}_2(\mathbb{Z})) \dashrightarrow \Phi_f : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$$

where for $\mathrm{GL}_2(\mathbb{A}) \ni g = \gamma g_\infty \kappa$, $g_\infty = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$,

$$\Phi_f(g) := (f|_k g_\infty)(i) := (\det g_\infty)^{k/2}(ci+d)^{-k} f\left(\frac{ai+b}{ci+d}\right).$$

- → Φ_f is a cuspidal automorphic form on $\mathrm{GL}_2(\mathbb{A})$.
- → $\mathrm{GL}_2(\mathbb{A})$ acts on Φ_f via $h \cdot \Phi_f(g) = \Phi_f(gh)$. This action gives rise to an automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A})$.

Similarly, for $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$:

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Classical Saito-Kurokawa lifting

Recall:

$$\begin{array}{ccccc} f & \longmapsto & & & SK(f) \\ \cap & & & & \cap \\ S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})) & \xrightarrow{\sim} & \text{Jacobi forms} & \longrightarrow & S_k(\mathrm{Sp}_4(\mathbb{Z})) \\ & & \text{(weight } k\text{, index 1, level 1)} & & \end{array}$$

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→ There exist two generalisations to $f \in S_k(\overbrace{\mathrm{SL}_2(\mathbb{Z}) \cap \left(\begin{smallmatrix} \mathbb{Z} & N\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)}^{\Gamma^0(N)}), N \in \mathbb{N}$, but hard to prove Maass relations directly from construction.

Representation theoretic Saito-Kurokawa lifting

$$\begin{array}{ccccc} \pi_f \text{ on } \mathrm{PGL}_2(\mathbb{A}) & \xleftarrow{\theta \text{ corr.}} & \widetilde{\mathrm{SL}_2}(\mathbb{A}) & \xrightarrow{\theta \text{ corr.}} & \Pi_F \text{ on } \mathrm{PGSp}_4(\mathbb{A}) \\ \uparrow & & & & \downarrow \\ f & & & & F \end{array}$$

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This means:

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Do such F satisfy Maass relations?

Remarks

F satisfies Maass relations $\overset{\text{Maass, '79}}{\Longleftrightarrow}$ F is classical SK lift $\overset{\text{W., 2009}}{\Longleftrightarrow}$ R-P conj. for F not true

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Evdokimov, 1980: F satisfies Maass relations iff $L(s, F)$ has a pole.

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Saito-Kurokawa lifts of higher levels

- → There is more than one lift from $f \in S_k(\Gamma^0(N))$. The number depends on $\#\{p : p|N\}$.

- → The representations Π_F are constructed locally from π_f and are all nearly equivalent.

- → Every Π_F is a so-called CAP representation, i.e. $\Pi_F = \otimes_{p\leq\infty}\Pi_p$ is equivalent at almost all places to a constituent of a globally induced representation from a proper parabolic subgroup of $\mathrm{GSp}_4(\mathbb{A})$ (in our case to $\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} \cap \mathrm{GSp}_4(\mathbb{A})$.

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Do F coming from P-CAP representations satisfy Maass relations?

Main result

For $N_1, N_2 \in \mathbb{N}$, $N_1 | N_2$ define

$$\Gamma_0(N_1, N_2) := \mathrm{Sp}_4(\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & N_1\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N_2\mathbb{Z} & N_2\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N_2\mathbb{Z} & N_2\mathbb{Z} & N_1\mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

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Theorem

Assume that $F \in S_k(\Gamma_0(N_1, N_2))$ is associated to a P-CAP representation. Then

$$a(F, L\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}) = \sum_{r | \gcd(a,b,c)} r^{k-1} a(F, L\begin{pmatrix} \frac{ac}{r^2} & \frac{b}{2r} \\ \frac{b}{2r} & 1 \end{pmatrix}),$$

where $\gcd(a, b, c, N_2) = 1$, $b^2 - 4ac < 0$, all prime factors of L divide N_2, and $\left(\frac{b^2-4ac}{p}\right) = -1$ for all $p | N_1$.

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Sketch of the proof

Write

$$F(Z) = \sum_T a(F, T) e^{2\pi i \operatorname{tr}(TZ)},$$

where T varies through $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$, $a, b, c \in \mathbb{Z}, \operatorname{disc} T := b^2 - 4ac < 0$.

$$F \in S_k(\Gamma_0(N_1, N_2)) \quad \Longrightarrow \quad \forall_{A \in \Gamma^0(N_1)} a(F, {}^tATA) = a(F, T).$$

We say: $T \sim T' \iff T' = {}^tATA$ for some $A \in \Gamma^0(N_1)$.
Let d be a negative fundamental discriminant, $\operatorname{cont} T := \gcd(a, b, c)$ and

$$H(dM^2, L; \Gamma^0(N_1)) := \{T : \operatorname{disc} T = dL^2M^2, \operatorname{cont} T = L\} / \sim .$$

We're interested in the representatives

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Why the representatives $L \begin{pmatrix} M & \\ & 1 \end{pmatrix} S_c \begin{pmatrix} M & \\ & 1 \end{pmatrix}, \; c \in \mathrm{Cl}_d(MN_1)$?

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- Evaluate B at $H(L, M)$, where

$$H(L,M)_p = \begin{cases} \begin{pmatrix} LM^2 & & & \\ & LM & & \\ & & 1 & \\ & & & M \end{pmatrix}, & p \mid LM \\ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, & p \nmid LM \text{ or } p = \infty \end{cases}$$

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- Relation between $B(H(L,M))$ and $B(H(L',M'))$, where $L'|L$, $M'|M$, and $\mathrm{ord}_p L = \mathrm{ord}_p L'$, $\mathrm{ord}_p M = \mathrm{ord}_p M'$ for every $p|N_2$.

Sketch of the proof

Theorem

Let $F \in S_k(\Gamma_0(N_1, N_2))$. Let $L, M, L', M' \in \mathbb{N}$ be such that $L'|L$, $M'|M$ and $\mathrm{ord}_p L = \mathrm{ord}_p L'$, $\mathrm{ord}_p M = \mathrm{ord}_p M'$ for every $p|N_2$. Then for all characters Λ of $\mathrm{Cl}_d(M'N_1)$:

$$(const.) \sum_{c \in \mathrm{Cl}_d(MN_1)} \Lambda^{-1}(c) a(F, L \begin{pmatrix} M & \\ & 1 \end{pmatrix} S_c \begin{pmatrix} M & \\ & 1 \end{pmatrix}) \prod_{\substack{p|L'M' \\ p\nmid N_2}} B_{\phi_p}(H_p(L', M'))$$

$$= \sum_{c' \in \mathrm{Cl}_d(M'N_1)} \Lambda^{-1}(c') a(F, L' \begin{pmatrix} M' & \\ & 1 \end{pmatrix} S_{c'} \begin{pmatrix} M' & \\ & 1 \end{pmatrix}) \prod_{\substack{p|LM \\ p\nmid N_2}} B_{\phi_p}(H_p(L, M)),$$

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Let $F \in S_k(\Gamma_0(N_1, N_2))$. Let $L, M, L', M' \in \mathbb{N}$ be such that $L'|L, M'|M$ and $\mathrm{ord}_p L = \mathrm{ord}_p L'$, $\mathrm{ord}_p M = \mathrm{ord}_p M'$ for every $p|N_2$. Then for all characters Λ of $\mathrm{Cl}_d(M'N_1)$:

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$$= \sum_{c' \in \mathrm{Cl}_d(M'N_1)} \Lambda^{-1}(c') a\left(F, L' \begin{pmatrix} M' & \\ & 1 \end{pmatrix} S_{c'} \begin{pmatrix} M' & \\ & 1 \end{pmatrix}\right) \prod_{\substack{p|LM \\ p\nmid N_2}} B_{\phi_p}(H_p(L, M)),$$

Sketch of the proof

Theorem

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Sketch of the proof

Theorem

Let $F \in S_k(\Gamma_0(N_1, N_2))$. Let $L, M, L', M' \in \mathbb{N}$ be such that $L'|L$, $M'|M$ and $\mathrm{ord}_p L = \mathrm{ord}_p L'$, $\mathrm{ord}_p M = \mathrm{ord}_p M'$ for every $p|N_2$. Then for all characters Λ of $\mathrm{Cl}_d(M'N_1)$:

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Maass relations:

- allowed to take $\Lambda = \mathbf{1}$;

Sketch of the proof

Theorem

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Maass relations:

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- Pitale, Saha, Schmidt: $B_{\phi_p}(H_p(L, M)) = \sum_{i=0}^{\mathrm{ord}_p(L)} B_{\phi_p}(H_p(1, LM/p^i))$.

Thank you for your attention!