

Maass relations for Saito-Kurokawa lifts of higher levels

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Satake, 1966: reformulation in terms of automorphic representations, generalisation to modular forms on other groups

Siegel modular forms of degree 2

$$\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{G}_k \quad f : \{z \in \mathbb{C} : \mathrm{Im} z > 0\} \rightarrow \mathbb{C}$$

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→ There exists a basis consisting of common eigenforms.

Ramanujan-Petersson conjecture

Definition

For a Siegel modular form F of degree 2, which is an eigenform with eigenvalues $\{\lambda_p, \lambda_{p^2} : p \text{ prime}\}$ the spinor L -function is given by

$$L(s, F) = \prod_p L_p(p^{-s})^{-1}, \quad s \in \mathbb{C},$$

where

$$\begin{aligned} L_p(X) &= \text{polynomial depending on } \lambda_p, \lambda_{p^2} \\ &= (1 - \alpha_{0,p}X)(1 - \alpha_{0,p}\alpha_{1,p}X)(1 - \alpha_{0,p}\alpha_{2,p}X)(1 - \alpha_{0,p}\alpha_{1,p}\alpha_{2,p}X). \end{aligned}$$

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Conjecture (Satake, 1966)

For $j = 1, 2$ and all prime numbers p :

$$|\alpha_{j,p}| = 1.$$

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Let $k \geq 10$ even. Then there exists an injective mapping

$$SK : S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})) \rightarrow S_k(\mathrm{Sp}_4(\mathbb{Z}))$$

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where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $L(s, f) = \sum_{n=1}^{\infty} \frac{a(f, n)}{n^s}$.

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(2 implies that $SK(f)$ does not satisfy Ramanujan-Petersson conjecture.)

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Resnikoff, Saldaña, 1974: discovery of F that satisfy the relation

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→ *Weissauer, 2009*: Ramanujan-Petersson conjecture holds for F that are not Saito-Kurokawa lifts.

$$\begin{array}{ccc} S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})) & \xrightarrow{\sim} & \text{Jacobi forms} & \longrightarrow & S_k(\mathrm{Sp}_4(\mathbb{Z})) \\ & & \text{(weight } k, \text{ index 1, level 1)} & & \\ & \uparrow & & & \uparrow \\ & \text{---} & & & \text{---} \\ \text{Eichler-Zagier corr. } \circ (\text{Shimura lift})^{-1} & & & & \text{due to Maass, 1979} \\ \text{(through modular forms of weight } k - 1/2) & & & & \end{array}$$

So far we considered only

$$SK: S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})) \rightarrow S_k(\mathrm{Sp}_4(\mathbb{Z})).$$

What about higher levels?

Representation theoretic prelude

Ring of adeles of \mathbb{Q} :

$$\mathbb{A} := \{(a_p)_{p \leq \infty} \in \prod_{p \leq \infty} \mathbb{Q}_p : a_p \in \mathbb{Z}_p \text{ for almost all } p\}.$$

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where

$$\mathrm{GSp}_{2n}(\mathbb{Q}) := \{g \in \mathrm{GL}_{2n}(\mathbb{Q}) : {}^t g \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix} g = \mu(g) \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}\}.$$

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Adelisation of modular forms:

$$f \in S_k(\mathrm{SL}_2(\mathbb{Z})) \quad \dashrightarrow \quad \Phi_f: \mathrm{GL}_2(\mathbb{A}) \rightarrow \mathbb{C}$$

where for $\mathrm{GL}_2(\mathbb{A}) \ni g = \gamma g_\infty \kappa$, $g_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\Phi_f(g) := (f|_k g_\infty)(i) := (\det g_\infty)^{k/2} (ci + d)^{-k} f\left(\frac{ai + b}{ci + d}\right).$$

→ Φ_f is a cuspidal automorphic form on $\mathrm{GL}_2(\mathbb{A})$.

→ $\mathrm{GL}_2(\mathbb{A})$ acts on Φ_f via $h \cdot \Phi_f(g) = \Phi_f(gh)$. This action gives rise to an automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A})$.

Similarly, for $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$:

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Classical Saito-Kurokawa lifting

Recall:

$$\begin{array}{ccc} \begin{array}{c} f \\ \cap \\ S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})) \end{array} & \begin{array}{c} \dashrightarrow \\ \xrightarrow{\sim} \end{array} & \begin{array}{c} SK(f) \\ \cap \\ S_k(\mathrm{Sp}_4(\mathbb{Z})) \end{array} \\ & & \text{Jacobi forms} \\ & & \text{(weight } k, \text{ index } 1, \text{ level } 1) \end{array}$$

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→ There exist two generalisations to $f \in S_k(\mathrm{SL}_2(\mathbb{Z}) \cap \overbrace{(\frac{\mathbb{Z}}{\mathbb{Z}} \frac{N\mathbb{Z}}{\mathbb{Z}})})$, $N \in \mathbb{N}$,
 but **hard to prove Maass relations directly from construction**.

Representation theoretic Saito-Kurokawa lifting

$$\begin{array}{ccccc} \pi_f \text{ on } \mathrm{PGL}_2(\mathbb{A}) & \xleftarrow{\theta \text{ corr.}} & \widetilde{\mathrm{SL}}_2(\mathbb{A}) & \xrightarrow{\theta \text{ corr.}} & \Pi_F \text{ on } \mathrm{PGSp}_4(\mathbb{A}) \\ \uparrow & & & & \downarrow \\ f & & & & F \end{array}$$

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This means:

For each eigenform $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ with k even there exists an eigenform $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ such that

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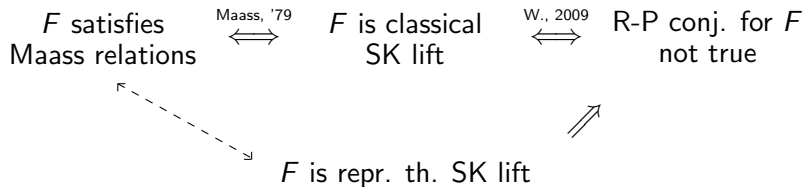
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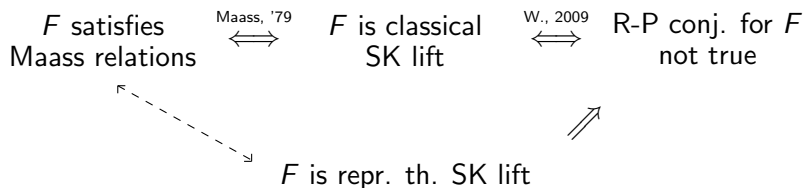
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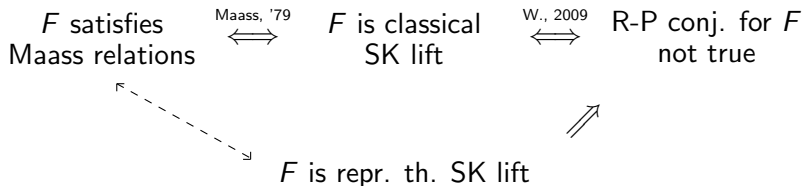
$$L(s, F) = \zeta(s - k + 2)\zeta(s - k + 1)L(s, f).$$

Do such F satisfy Maass relations?



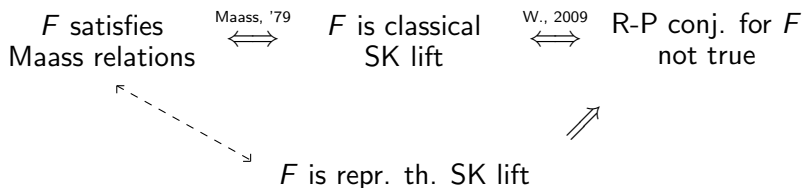


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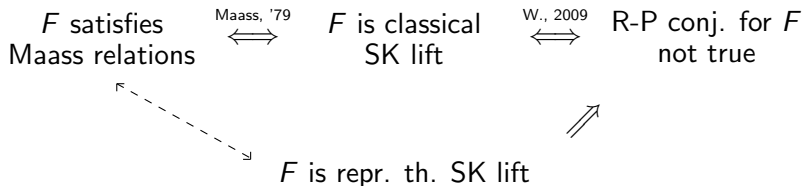


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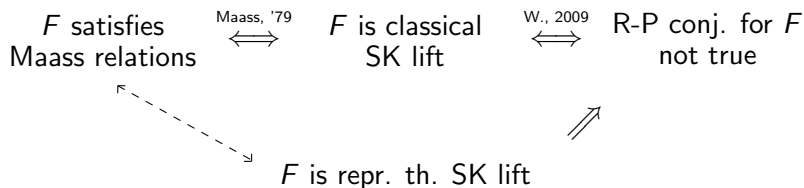
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→ Higher level?



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→ Higher level? Generalisation of the second approach.

Saito-Kurokawa lifts of higher levels

- There is more than one lift from $f \in S_k(\Gamma^0(N))$. The number depends on $\#\{p : p|N\}$.
- The representations Π_F are constructed locally from π_f and are all nearly equivalent.
- Every Π_F is a so-called CAP representation, i.e. $\Pi_F = \otimes_{p \leq \infty} \Pi_p$ is equivalent at almost all places to a constituent of a globally induced representation from a proper parabolic subgroup of $\mathrm{GSp}_4(\mathbb{A})$ (in our case to $\left(\begin{smallmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{smallmatrix} \right) \cap \mathrm{GSp}_4(\mathbb{A})$).
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Do F coming from P-CAP representations satisfy Maass relations?

Main result

For $N_1, N_2 \in \mathbb{N}$, $N_1 | N_2$ define

$$\Gamma_0(N_1, N_2) := \mathrm{Sp}_4(\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & N_1\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N_2\mathbb{Z} & N_2\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N_2\mathbb{Z} & N_2\mathbb{Z} & N_1\mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

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Theorem

Assume that $F \in S_k(\Gamma_0(N_1, N_2))$ is associated to a P -CAP representation.

Then

$$a(F, L\left(\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\right)) = \sum_{r | \mathrm{gcd}(a, b, c)} r^{k-1} a(F, L\left(\begin{pmatrix} \frac{ac}{r^2} & \frac{b}{2r} \\ \frac{b}{2r} & 1 \end{pmatrix}\right)),$$

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Sketch of the proof

Write

$$F(Z) = \sum_T a(F, T) e^{2\pi i \operatorname{tr}(TZ)},$$

where T varies through $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$, $a, b, c \in \mathbb{Z}$, $\operatorname{disc} T := b^2 - 4ac < 0$.

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Why the representatives $L \begin{pmatrix} M & \\ & 1 \end{pmatrix} S_c \begin{pmatrix} M & \\ & 1 \end{pmatrix}$, $c \in \text{Cl}_d(MN_1)$?

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Π irr. aut.
repr. of $\mathrm{GSp}_4(\mathbb{A})$



Bessel model
(of type (Λ, θ))

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$\xrightarrow{\sim}$

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Sketch of the proof

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- Evaluate B at $H(L, M)$, where

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- Relation between $B(H(L, M))$ and $B(H(L', M'))$, where $L' | L, M' | M$, and $\mathrm{ord}_p L = \mathrm{ord}_p L', \mathrm{ord}_p M = \mathrm{ord}_p M'$ for every $p | N_2$.

Sketch of the proof

Theorem

Let $F \in S_k(\Gamma_0(N_1, N_2))$. Let $L, M, L', M' \in \mathbb{N}$ be such that $L'|L, M'|M$ and $\text{ord}_p L = \text{ord}_p L', \text{ord}_p M = \text{ord}_p M'$ for every $p|N_2$. Then for all characters Λ of $\text{Cl}_d(M'N_1)$:

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- Pitale, Saha, Schmidt: $B_{\phi_p}(H_p(L, M)) = \sum_{i=0}^{\text{ord}_p(L)} B_{\phi_p}(H_p(1, LM/p^i))$.

Thank you for your attention!