On Bessel models for GSp_4 and Fourier coefficients of Siegel modular forms of degree 2

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To my parents.

Abstract

In this work, we make a detailed study of the Fourier coefficients of cuspidal Siegel modular forms of degree 2. We derive a very general relation between the Fourier coefficients that extends previous work in this direction by Andrianov, Kowalski-Saha-Tsimerman and others. The basis for our relation is the dependence between values of global Bessel periods and averages of Fourier coefficients. Consequently our relation applies also to Bessel periods of more general automorphic forms on $\mathrm{GSp}_4(\mathbf{A})$.

We use our relation to prove that cuspidal Siegel modular forms associated to P-CAP representations (Saito-Kurokawa lifts with level) satisfy the so-called Maass relations. This is the first result of this kind for Siegel modular forms with respect to general congruence subgroups. Another important corollary of our work is the existence of non-zero Fourier coefficients of the simplest form possible (often fundamental or primitive) for a wide family of cuspidal Siegel modular forms of degree 2.

Finally, using classical methods, we are able to prove that paramodular newforms of square-free level have infinitely many non-zero fundamental Fourier coefficients. This result extends previous work by Saha in the full-level case, and is especially interesting because of the paramodular conjecture connecting paramodular newforms of weight 2 and rational abelian surfaces.

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Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Any views expressed in the dissertation are those of the author.

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Chapter 1

Introduction

This work is devoted to certain families of modular forms and their various properties. Modular forms are one of those incredible objects that occur almost throughout the whole of mathematics and join seemingly very different areas. We begin with some introductory sections. Our main results are described in Section 1.4.

1.1 Beginnings of the theory of modular forms

Let Q be a quadratic form over $\mathbf{Z}$ in k variables, e.g.

$$Q(x_1, \ldots, x_k) = a_1 x_1^2 + \ldots + a_k x_k^2 \, .$$

Some of the most natural questions one might ask about Q are the following: For which $n \in \mathbf{N}$ does $Q(x_1, \ldots, x_k) = n$ have a solution in $\mathbf{Z}^k$? What is the minimal k we could take? Or, more generally, what is the number of representations of n by Q,

$$r_Q(n) := \#\{(x_1, \ldots, x_k) \in \mathbf{Z}^k : Q(x_1, \ldots, x_k) = n\} \, ?$$

These kinds of questions drove Jacobi in the first half of the 19th century to introduce the *theta function*

$$\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q = e^{2\pi i z}, z \in \mathbf{C} \text{ with } \operatorname{Im} z > 0 \, ,$$

as $\theta^k(z) = \sum_{n\geq 0} r_k(n)q^n$, where $r_k(n) := r_Q(n)$ for $Q(x_1,\ldots,x_k) = x_1^2 + \ldots + x_k^2 = n$. In general, one associates to Q a theta series

$$\Theta_Q(z) = \sum_{(x_1,\ldots,x_k)\in\mathbf{Z}^k} q^{Q(x_1,\ldots,x_k)} = \sum_{n\geq 0} r_Q(n)q^n \,.$$

Both θ and Θ_Q are examples of *modular forms of half-integral weight.* For classical results on r_Q see for example [5], and for more detailed survey [56].

However, the term "modular form", or "Modulform", apparently introduced by Klein, appears for the first time in the work of Hecke in 1924, [11]. The adjective "modular" comes from *modulus*, one of invariants of elliptic curves ([10]). Studies of isogenous elliptic curves and their modulus led Eisenstein, Kronecker, Klein and other mathematicians of 19th century to a discovery of modular curves, and from there to modular functions and modular forms.

Modular forms are holomorphic complex valued functions defined on the complex upper half-plane

$$\mathcal{H}_1 = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}\,,$$

that are invariant under the action of $\mathrm{SL}_2(\mathbf{Z})$ or some congruence subgroup Γ, and satisfy certain boundary conditions. This is enough to prove that they admit a Fourier expansion,

$$f(z) = \sum_{n=0}^{\infty} a(f,n)e^{2\pi i n z}\,.$$

One of the most fundamental *cusp forms*, modular forms that vanish at cusps of $\mathrm{SL}_2(\mathbf{Z})$ or Γ, is the *modular discriminant*

$$\Delta(z) = (2\pi)^{12}\sum_{n=1}^{\infty}\tau(n)e^{2\pi i n z} = (2\pi)^{12}e^{2\pi i z}\prod_{n=1}^{\infty}(1-e^{2\pi i n z})^{24}\,,$$

which is very closely related to the discriminant of an elliptic curve (cf. [58]). It was extensively studied by Ramanujan, who noticed, among others, that its Fourier coefficients are multiplicative. This fact was proven by Mordell, and then generalised by Hecke. Hecke observed that the Fourier coefficients $a(f,n)$ of some modular forms, normalised so that $a(f,1) = 1$, could be interpreted as eigenvalues of certain averaging operators $T(n)$ defined on the space of modular forms, now called *Hecke operators.* The multiplicativity of those implies multiplicativity of Fourier coefficients of

Hecke eigenforms. A bit later, when Petersson introduced an inner product on the space of cusp forms, it became clear that this space, classified according to two parameters: weight k and level N, possesses a basis of eigenforms for the Hecke operators $T(n)$ with n coprime to the level N. For fixed N and k such a basis is finite.

Together with development of this theory, more and more connections with other areas of mathematics were observed. Hilbert, inspired by Kronecker's "Jugendtraum", defined and explored modular forms invariant under the action of the group $\mathrm{SL}_2(\mathcal{O}_K)$ with $\mathcal{O}_K$ the ring of integers of a totally real field. Siegel went in another direction. Having been interested in the number of representations of quadratic forms by integral quadratic forms, i.e.

$$R_Q(T) := \#\{X \in \mathbf{Z}^{m\times n} \colon {}^tXQX = T\}\,,$$

he generalised the idea of Jacobi and introduced the *Siegel upper half space*[1]

$$\mathcal{H}_n = \{Z \in \mathbf{C}^{n\times n} : Z = Z^t, \operatorname{Im} Z \text{ positive definite}\}$$

and a theta series of degree n

$$\Theta_Q^{(n)}(Z) = \sum_{X\in\mathbf{Z}^{m\times n}} e(\operatorname{tr}(X^tQXZ)) = \sum_T R_Q(T)e(\operatorname{tr}(TZ))$$

defined on $\mathcal{H}_n$, where $e(x) = e^{2\pi i x}$, and $Q \in \frac{1}{2}\mathbf{Z}^{m\times m}$, $T \in \frac{1}{2}\mathbf{Z}^{n\times n}$ are positive semidefinite symmetric matrices with integers on the diagonal. $\Theta_Q^{(n)}$ is an example of a *Siegel modular form of degree* n and weight $m/2$. Once again, Fourier coefficients carry important arithmetic information! For results on R_Q see [56].

1.2 Fourier coefficients

We will properly define Siegel modular forms in a later chapter. To motivate our research it will be enough to know that they are holomorphic complex valued functions on $\mathcal{H}_n$ that are invariant under the action of $\mathrm{Sp}_{2n}(\mathbf{Z})$ or

[1] This space is not as artificial as it may seem. It can be thought of as a set of all period matrices of a Riemann surface of genus n, another subject of Siegel's interests. Because of this, $\mathcal{H}_n$ is also called the Siegel upper half plane of genus n, and the words "degree" and "genus" related to the objects defined on it tend to be used interchangeably.

its congruence subgroups, and therefore admit a Fourier expansion[2]

$$F(Z) = \sum_{\substack{T=T^t, T\geq 0\\ \text{half-integral}}} a(F,T) e(\mathrm{tr}\,(TZ)),$$

where $T \in \frac{1}{2}\mathbf{Z}^{n\times n}$ positive semidefinite with an integral diagonal. Already looking at the Fourier expansion we can see that this more general situation is much more complicated. In the classical situation, when $n=1$, Fourier coefficients are fairly well understood. The space of modular forms has a basis consisting of Hecke eigenforms, whose coefficients are multiplicative and can be identified with the eigenvalues of these operators. This immediately tells us that non-zero cusp forms that are eigenfunctions of the Hecke operators must have their first coefficient non-zero. In the case $n > 1$, we also have a good notion of Hecke operators which are normal with respect to the Petersson inner product, and so their common eigenforms constitute a basis of Siegel modular forms. *This time, however, the Fourier coefficients are much more mysterious and the knowledge on Hecke eigenvalues cannot guarantee their understanding.* This happens already in the case of modular forms of half-integral weight[3], where we can only compare the coefficients $a(f,m)$ with m differing by a square of a prime. We face a very similar situation when $n=2$. We may vary the Fourier coefficients $a(F,T)$ in two ways: vertical, where the discriminant of the matrix T varies only by a square, that is $\operatorname{disc} T \in d(\mathbf{Z}^+)^2$ and a fundamental discriminant d is fixed; or horizontal, where d itself varies. *The action of Hecke operators only provides a relation between Fourier coefficients belonging to the same vertical class.* However, it is important to have an information about a single coefficient $a(f,m)$ or $a(F,T)$, especially with m or $\operatorname{disc} T$ square-free. These Fourier coefficients carry important arithmetic information - the square of their averages, when taken over a fixed discriminant, is proportional to a central L-value. This link was first observed by Böcherer [4], and then widely generalised - but proven in some special cases - by Furusawa, Martin, Shalika, Gan, Gross, Prasad, Takloo-Bighash, Ryan, Tornaria and others ([15], [16], [42], [47], [17], ...).

In this work we are occupied with the problem of characterising non-zero Siegel modular forms of degree 2 by their Fourier coefficients. We often

[2]The boundary condition is implied immediately for $n > 1$ (Koecher's theorem, [22]).

[3]In general, Hecke eigenforms of half-integral weight are not determined by their Hecke eigenvalues (see for example [43], Example 3.4.7). However, if they belong to the so-called Kohnen plus new space and have level $4N$, where N is odd and square-free, it suffices to know almost all Hecke eigenvalues, i.e. strong multiplicity one holds ([24]).

restrict ourselves to eigenfunctions of Hecke operators[4], and call them in short Hecke eigenforms. One can take a strictly computational point of view and be interested in the smallest set of Fourier coefficients of F that asserts that $F = 0$. To make this question simpler one can look at the coefficients modulo a prime p. In this setting the bound was first given by Sturm [59] in 1984 for modular forms of degree 1. A bound for Siegel modular forms of degree 2 was obtained much later, in [41] and [6], and only in 2015 Richter and Raum [44] with a completely different method provided a bound for a general degree n. All these results, however, rely on certain conditions that do not cover all the cases. It is worth to notice here that when $n > 1$, it is not at all clear how one should order the matrices in the Fourier expansion and which quantity should be bounded. The aforementioned results bound the diagonal entries of all those matrices.[5]

Our point of view is motivated by the great importance of an infinite subset of Fourier coefficients, *fundamental Fourier coefficients*, in the theory of Bessel models and L-functions. These are the coefficients $a(F,T)$ for which the discriminant of T is a fundamental discriminant. Their important role manifested first in a paper of Furusawa [14], who under an assumption of non-vanishing of such a coefficient obtained a special value result for an L-function for $\mathrm{GSp}_4 \times \mathrm{GL}_2$; later works on this topic built on Furusawa's result and proved analytic and arithmetic properties of various related L-functions (see [49] for more details). At the same time fundamental Fourier coefficients seem to determine Siegel modular forms. The first result in this direction was given by Saha [51] who proved that a non-zero cusp form invariant under the action of $\mathrm{Sp}_4(\mathbf{Z})$ is indeed determined by its fundamental Fourier coefficients. If we look at the cusp forms that are invariant only under the action of a smaller subgroup Γ of $\mathrm{Sp}_4(\mathbf{Z})$, the situation becomes much more complicated. Firstly, there are many groups Γ that might be of interest, and secondly, the case when the level N is divisible by a square provides many obstacles. The only known results concern Siegel congruence subgroup of $\mathrm{Sp}_4(\mathbf{Z})$ of square-free level N ([52], [49] or Theorem 5.2.2). I obtain a similar result (Theorem 5.3.1) for an important class of Siegel modular forms that are invariant under the action of the paramodular subgroup - *paramodular forms*; more on this in Section 5.2, 4.4. The significance of paramodular forms comes from their connection with abelian surfaces (paramodular conjecture) via the equality of corresponding L-functions.

[4]We specify these Hecke operators in later chapters.
[5]This is still a finite set, because $\det T$ has to be non-negative.

The non-vanishing result for fundamental Fourier coefficients of paramodular forms mentioned above requires several ingredients for its proof, the first of which is the existence of a non-vanishing primitive Fourier coefficient. This can be done using classical methods relating to the action of Hecke operators on Fourier coefficients; however, a more natural way to approach such problems is via the theory of Bessel models. This latter approach relies crucially on interpreting modular forms and their Fourier coefficients from a representation-theoretic point of view, which we describe next.

1.3 Automorphic forms and representations

A stepping stone in the theory of modular forms was the discovery, first by Gelfand and Fomin, that modular forms can be thought of as smooth vectors of representations of a Lie group G on spaces of certain holomorphic functions on G that are left-invariant under a discrete subgroup Γ of G - *automorphic forms*. Later, localising this idea to all the completions of $\mathbf{Q}$ led to automorphic forms defined on the group $G(\mathbf{A})$ of adelic points of G. The right regular action of $G(\mathbf{A})$ on these functions gives rise to automorphic representations of $G(\mathbf{A})$. In the case when $G = \mathrm{GL}_n$ all such representations are generic, and thus admit a Whittaker model. Vectors in this model are basically Fourier coefficients of the automorphic forms on $\mathrm{GL}_n(\mathbf{A})$. The simplicity and uniqueness of this model is extremely useful for studying cuspidal automorphic forms on $\mathrm{GL}_n(\mathbf{A})$. Their space decomposes into a direct sum of vector spaces of irreducible automorphic representations. The celebrated multiplicity one result for $\mathrm{GL}_n(\mathbf{A})$ states that each such representation occurs at most once.[6] Moreover, two cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A})$ are isomorphic if their local components are isomorphic for all but a finite number of places.[7]

In the case when $G = \mathrm{GSp}_{2n}$, the representations at the infinite place associated to holomorphic Siegel modular forms are not generic, and therefore a Whittaker model does not exist. A good substitute in the case $n = 2$ is a Bessel model ([36]). Such a model always exists ([26]) and, as we shall see in Section 4.3, values of its vectors, Bessel periods, are also related to Fourier coefficients. Curiously, if we focus on representations coming

[6] This statement was proven by Jacquet, Langlands [20] for $n = 2$ and independently by Piatetski-Shapiro [32] and Shalika [57] for $n > 2$.

[7] The last statement is known as *strong multiplicity one* theorem, proven by Piatetski-Shapiro [32] and Jacquet, Shalika [21].

from Siegel modular forms of degree 2, the existence of a non-zero fundamental Fourier coefficient implies that multiplicity one for GSp_4 follows from Böcherer's conjecture (cf. [48]). In this setting the multiplicity one conjecture may be stated in terms of Hecke eigenvalues, i.e. if two Hecke eigenforms have the same eigenvalues for the operators $T(p)$ and $T(p^2)$ for all primes p, then they must be proportional.

This is only one of the instances when switching between a representation theoretic structure and a classical language gives an insight into a classical theory. Another important example of this interplay is the celebrated modularity theorem, which provides a relation - via the equality of L-functions - between Hecke eigenvalues of certain modular forms and the number of points on suitable elliptic curves over finite fields. This may be also viewed as a special case of the Langlands functoriality conjecture.

Our work provides many other examples when using the representation theoretic structure of number theoretic objects and switching between these two worlds turned out to be very beneficial.

1.4 Main results

Henceforth we focus on Siegel cusp forms F of degree 2 ($n = 2$). One of the main aims of this thesis was to assure the existence of non-zero Fourier coefficients $a(F, T)$ with discriminant of T simplest possible for a wide family of Siegel modular forms of degree 2. In order to do that, we chose to work with Siegel modular forms that are invariant under the action of the group

$$\Gamma_0(N_1, N_2) := \mathrm{Sp}_4(\mathbf{Z}) \cap \begin{pmatrix} \mathbf{Z} & N_1\mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ N_2\mathbf{Z} & N_2\mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ N_2\mathbf{Z} & N_2\mathbf{Z} & N_1\mathbf{Z} & \mathbf{Z} \end{pmatrix}, \quad N_1 | N_2.$$

This includes paramodular forms and Siegel modular forms invariant under the action of the Siegel or Borel congruence subgroups of $\mathrm{Sp}_4(\mathbf{Z})$. Looking at this invariance property, it is easy to see that the Fourier coefficients of such a form F satisfy the equality

$$a(F, T) = a(F, {}^tATA) \qquad \text{for all} \quad A \in \Gamma^0(N_1).$$

If $T' = {}^tATA$ for some $A \in \Gamma^0(N_1)$, we say that the matrices T, T' (or the corresponding Fourier coefficients) are $\Gamma^0(N_1)$-*equivalent*; we introduce the set $H(dM^2, L; \Gamma^0(N_1))$ consisting of all $\Gamma^0(N_1)$-equivalence classes of matrices T whose discriminant is equal to dM^2L^2, T/L is a primitive matrix (such L is called a *content* of T) and d is a (negative) fundamental discriminant. Equivalently,

$$H(dM^2, L; \Gamma^0(N_1)) = \{ \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} : \begin{array}{c} a, b, c \in \mathbf{Z},\ \gcd(a, b, c) = L, \\ b^2 - 4ac = dM^2L^2 \end{array} \} / \sim,$$

where $T \sim T'$ means that there exists $A \in \Gamma^0(N_1)$ such that $T' = {}^tATA$.

It is known ([35]) that if $N_1 = 1$, the elements of $H(dM^2, L; \Gamma^0(1))$ are in a bijective correspondence with the elements of a ray class group $\mathrm{Cl}_d(M)$ of $\mathbf{Q}(\sqrt{d})$ ($\mathrm{Cl}_d(1)$ is simply the ideal class group of $\mathbf{Q}(\sqrt{d})$). If $N_1 > 1$, the ray class group $\mathrm{Cl}_d(MN_1)$ is in general smaller than $H(dM^2, L; \Gamma^0(N_1))$ (cf. Chapter 3). In any case, the set of $\Gamma^0(N_1)$-equivalence classes of Fourier coefficients is finite.

Hence, it would be useful if we could provide a relation between Fourier coefficients $a(F, T)$ that are supported on the representatives of the sets $H(dM^2, L; \Gamma^0(N_1))$ and $H(dM'^2, L'; \Gamma^0(N_1))$, where $M'|M$, $L'|L$ are smallest possible (ideally $M' = L' = 1$). This is the subject of Sections 4.2, 4.3. We describe a simple case now.

1.4.1 Relations among Fourier coefficients

Our main result on this topic may be simplified to the following form:

Theorem. *Let F be a cuspidal Siegel modular form of degree 2, level $\Gamma_0(N_1, N_2)$ and weight k. Suppose that F is an eigenform of the local Hecke algebra at all primes $p \nmid N_2$. Let d be a negative fundamental discriminant and let L, M, L', M' be positive integers such that*

$$L'|L, \quad M'|M, \quad (L, N_2^\infty) = (L', N_2^\infty), \quad (M, N_2^\infty) = (M', N_2^\infty)$$

(cf. Section 1.5). Assume moreover that $\left(\frac{dM^2}{p}\right) = -1$ for all primes $p|N_1$. Then for all characters Λ of $H(dM'^2, L', \Gamma^0(N_1)) \cong \mathrm{Cl}_d(M'N_1)$,

$$\frac{|\mathrm{Cl}_d(M'N_1)|}{|\mathrm{Cl}_d(MN_1)|} \left(\frac{L'M'}{LM}\right)^k \sum_{T \in H(dM^2, L; \Gamma^0(N_1))} \Lambda^{-1}(T) a(F, T) B(L', M')$$
$$= \sum_{T' \in H(dM'^2, L'; \Gamma^0(N_1))} \Lambda^{-1}(T') a(F, T') B(L, M), \quad (*)$$

where the completely explicit function B (depending on Λ and the Hecke *eigenvalues of F) can be found using Sugano's Theorem 2.2.2.*

Remark. In the above theorem we implicitly use the fact that there is a natural surjective map from $H(dM^2, L; \Gamma^0(N_1))$ to $H(dM'^2, L'; \Gamma^0(N_1))$ whenever $M'|M$.

Remark. The condition $\left(\frac{dM^2}{p}\right) = -1$ ensures that $H(dM^2, L; \Gamma^0(N_1))$ has a natural *group structure*. We actually prove a more general result without this assumption. Then the sum runs over a subset $H_1(dM^2, L; \Gamma^0(N_1))$ of $H(dM^2, L; \Gamma^0(N_1))$ that possesses a group structure; more on this in Chapter 3.

Remark. In fact, we obtain much more general result (Theorem 4.2.1 or Corollary 4.2.1) that is applicable to arbitrary automorphic forms on $\mathrm{GSp}_4(\mathbf{A})$ (right invariant under I_{N_2} or I_{N_1,N_2}) and does not require the conditions $(L, N_2^\infty) = (L', N_2^\infty)$, $(M, N_2^\infty) = (M', N_2^\infty)$. However, Theorem 4.2.1 includes ramified terms which have been explicitly computed only in certain cases, and that forces us to put additional assumptions. The main issue is non-vanishing of certain values of vectors in local Bessel models at primes $p|N_2$.

This theorem improves the relation obtained by Andrianov [2] and Kowalski, Saha, Tsimerman [25] to Siegel modular forms that are invariant under $\Gamma_0(N_1, N_2)$ with $N_1 > 1$. The last assumption on dM^2 may be omitted, but then the sum runs over a proper subset of $H(dM^2, L; \Gamma^0(N_1))$ (cf. Section 3.3), and therefore does not include all the coefficients with given content and discriminant. Our more general Theorem 4.2.1 includes the information at ramified places and is the first result of this type written down.

The key ingredients to prove this theorem are

- the fact that certain values of global Bessel periods may be expressed in terms of averages of Fourier coefficients of suitable automorphic forms,
- a relation (4.4) between local and global Bessel functionals.

We carry out the necessary calculations in Sections 4.2, 4.3. Stating the equality (4.4) outside a finite set of places gives us more flexibility in formulating our results.

1.4.2 Non-vanishing of 'simple' Fourier coefficients

The theorem presented above allows us to deduce a lot of information on Fourier coefficients. In particular, we may assure the existence of a 'simple'

Fourier coefficient.

Theorem. *Let F be a non-zero cuspidal Siegel modular form of degree 2, level $\Gamma_0(N_1, N_2)$ and weight k. Assume that F is an eigenform of the local Hecke algebra at all primes $p \nmid N_2$. Let d be a fundamental discriminant and let L, M be positive integers such that there exists a matrix T_0 with $a(F, T_0) \neq 0$ and $\operatorname{disc} T_0 = dM^2L^2$, $\operatorname{cont} T_0 = L$. Assume that $\left(\frac{dM^2}{p}\right) = -1$ for all primes $p|N_1$. Then there exists a matrix T with content equal to (L, N_2^∞) such that $a(F, T) \neq 0$. In particular, if $\gcd(L, N_2) = 1$, $a(F, T) \neq 0$ for a primitive matrix T.*

Remark. In the special case $N_1 = 1$ (i.e. F is a cusp form with respect to the Siegel congruence subgroup $\Gamma_0^{(2)}(N_2)$), the condition $\left(\frac{dM^2}{p}\right) = -1$ for all primes $p|N_1$ is trivially true. Hence our theorem implies the existence of a non-vanishing coefficient whose content only contains primes dividing N_2. This was also proved by Yamana in [62].

The reason we cannot deduce $a(F, T) \neq 0$ for a primitive matrix T in general from the relation $(*)$ is because we forced it to exclude the information on B_{ϕ_p} at primes $p|N_2$. These B_{ϕ_p} are vectors in a local Bessel model for a representation π that is uniquely associated with F. Computing the values of B_{ϕ_p} explicitly at $p|N_2$ is in general complicated. However, in the cases when it was done (e.g. [34], [38]), we can improve our theorem and deduce non-vanishing of primitive Fourier coefficients. This is an example of a beneficial interplay between number theory and representation theory.

Theorem. *Let F be a non-zero cuspidal Siegel modular form of degree 2, level $\Gamma_0(N_1, N_2)$ with N_1, N_2 square-free, and weight k. Assume that F is an eigenform of the local Hecke algebra at all primes p, and let $\pi = \otimes \pi_p$ be the representation associated to F. Let d be a fundamental discriminant and let L, M be positive integers such that $a(F, T_0) \neq 0$ and $\operatorname{disc} T_0 = dM^2L^2$, $\operatorname{cont} T_0 = L$. Assume that $\left(\frac{dM^2}{p}\right) = -1$ for all primes $p|N_1$. Let Λ be such that*

$$\sum_{T \in H(dM^2, L; \Gamma^0(N_1))} \Lambda^{-1}(T) a(F, T) \neq 0\,.$$

Under some mild technical assumptions on π_p at $p|N_2$ (cf. Corollary 4.4.2), there exists a primitive matrix T such that $a(F, T) \neq 0$.

1.4.3 Non-vanishing of fundamental Fourier coefficients of paramodular forms

The theorems mentioned in the previous subsection prove non-vanishing of 'simple' Fourier coefficients (i.e., those whose content or discriminant is small or can be controlled) under the hypothesis that there exists a matrix T_0 with $a(F,T_0) \neq 0$ and $disc(T) = dM^2L^2$ with $\left(\frac{dM^2}{p}\right) = -1$ for all primes p dividing N_1. Such a seed coefficient can be shown to exist in many cases (e.g., it trivially exists if $N_1 = 1$), but an important case when such a coefficient does not exist is the case of paramodular forms.

Paramodular forms are examples of Siegel modular forms that are invariant under the action of the paramodular subgroup

$$\Gamma^{\text{para}}(N) := \mathrm{Sp}_4(\mathbf{Q}) \cap \begin{pmatrix} \mathbf{Z} & N\mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \mathbf{Z}/N \\ \mathbf{Z} & N\mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ N\mathbf{Z} & N\mathbf{Z} & N\mathbf{Z} & \mathbf{Z} \end{pmatrix}$$

for some $N \in \mathbf{N}$. Note that the group $\Gamma^{\text{para}}(N)$ contains $\Gamma_0(N,N)$, and thus in principle the aforementioned results concern also paramodular forms. However, it may be easily shown (cf. Section 5.1.1) that if F is invariant under the action of $\Gamma^{\text{para}}(N)$, then necessarily $a(F,T) = 0$ for every matrix $T \in H(dM^2, L; \Gamma^0(N))$ such that $\left(\frac{dM^2}{p}\right) = -1$ for any $p|N$. In other words, the condition we need (even its weaker version) never holds and hence a seed coefficient as above does not exist. This makes the results of the previous subsection not applicable to paramodular forms directly.

Nonetheless, we are able to prove the following result for square-free N, which is in fact far stronger than the results of the previous subsection as we are able to get all the way down to fundamental Fourier coefficients.

Theorem. *Let F be a non-zero paramodular newform of square-free level N and even weight $k \geq 2$. Then F has infinitely many non-zero fundamental Fourier coefficients.*

The above theorem is of deep significance because paramodular newforms play a key part in the higher dimensional analogue of the modularity theorem, known as the paramodular conjecture. As mentioned above, we cannot deduce it directly from the relation $(*)$. Indeed, we use ramified Hecke operators, in the classical language, to first prove the existence of a primitive Fourier coefficient. Once this is done, the existence of a fundamental Fourier coefficient requires moving into the world of Jacobi forms and half integral weight forms, following a strategy used previously by Saha [49] and Saha, Schmidt [52]. All this is carried out in Chapter 5.

1.4.4 Maass relations

It is known that a Siegel modular form F is a (classical) Saito-Kurokawa lift of an elliptic modular form f if and only if its Fourier coefficients satisfy the *Maass relations*

$$a(F, \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}) = \sum_{r \mid \gcd(a,b,c)} r^{k-1} a(F, \begin{pmatrix} \frac{ac}{r^2} & \frac{b}{2r} \\ \frac{b}{2r} & 1 \end{pmatrix}).$$

The classical cuspidal Saito-Kurokawa lift of weight k is a lift from a cuspidal modular form $f \in S^{(1)}_{2k-2}(\mathrm{SL}_2(\mathbf{Z}))$ with k even; it is a cuspidal Siegel modular form $F \in S^{(2)}_k(\mathrm{Sp}_4(\mathbf{Z}))$. The first construction of such a lift was given by Maass in [28] using correspondences between Siegel and classical modular forms, Jacobi forms and modular forms of half-integral weight (see also [12]). However, Saito-Kurokawa lifts can be also constructed using representation theory ([33], [55]). The advantage of the latter is that it can be easily generalised to lifts of modular forms of higher level, and also with an odd weight. In this case, if k is even[8], for any $f \in S^{(1)}_{2k-2}(\Gamma_0(N))$ we get a cuspidal Siegel modular form of weight k invariant under the action of a congruence subgroup of $\mathrm{GSp}_4(\mathbf{Z})$ such that its spin L-function is given by

$$L(s,F) = L(s,f)\zeta(s-k+1)\zeta(s-k+2).$$

This does not tell us though anything about the coefficients of F and whether they satisfy similar Maass relations. Pitale, Saha and Schmidt [35] showed that this is indeed the case if $F \in S^{(2)}_k(\mathrm{Sp}_4(\mathbf{Z}))$ is a Hecke eigenform.

From a representation theoretic point of view, a Saito-Kurokawa lift produces from a cuspidal automorphic representation π of $\mathrm{PGL}_2(\mathbf{A})$ a cuspidal automorphic representation Π of $\mathrm{PGSp}_4(\mathbf{A})$; and we can think of f as an element in the vector space of π, and F a vector of matching weight in the vector space of Π. What is important is that any representation Π we obtain via this (generalised) Saito-Kurokawa lifting is a CAP representation.

More precisely, consider a cuspidal Siegel modular form F of level $\Gamma_0(N_1, N_2)$. We say that F is associated to a CAP representation if the

[8] If k is odd, the construction leads to a non-holomorphic function (cf. [30]).

following are true.

1) The adelisation of F gives rise to an irreducible automorophic representation π of $\mathrm{GSp}_4(\mathbf{A})$.

2) The representation π is equivalent at almost all places to a constituent of a globally induced representation from a proper parabolic subgroup of GSp_4.

Furthermore, we say that F is associated to a P-CAP representation if the proper parabolic subgroup above is the Siegel parabolic subgroup. The classical Saito-Kurokawa lifts correspond exactly to the P-CAP representations. It is known that if $k \geq 3$, then F that is associated to a CAP representation is automatically associated to a P-CAP representation. If $k = 1$ or 2, one also has CAP representations associated to other parabolics (the so-called B-CAP and Q-CAP representations).

Note that the first condition above automatically implies that F is an eigenform of the local Hecke algebra at all primes not dividing N_2. For general N_1, N_2, there is no known explicit construction that generalises the classical Saito-Kurokawa lifts and exhausts the set of all P-CAP F of level $\Gamma_0(N_1, N_2)$. It seems difficult then to directly prove the Maass relations from construction. In this work we are able to prove the Maass relations using methods of representation theory.

Theorem. *Let N_1, N_2 be positive integers and F be a cuspidal Siegel modular form of weight k and level $\Gamma_0(N_1, N_2)$ that is associated to a P-CAP representation. Let a, b, c be integers such that $\gcd(a, b, c, N_2) = 1$, $b^2 - 4ac < 0$ and $\left(\frac{b^2-4ac}{p}\right) = -1$ for all $p|N_1$. Let L be any positive integer dividing N_2^∞ (i.e., all prime factors of L divide N_2). Then*

$$a(F, L\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}) = \sum_{r|\gcd(a,b,c)} r^{k-1} a(F, L\begin{pmatrix} \frac{a}{r^2} & \frac{b}{2r} \\ \frac{b}{2r} & 1 \end{pmatrix}).$$

This theorem is another consequence of the relation $(*)$. It is an extension of the result of [35] to the lifts from modular forms of higher levels.

1.5 Notation

- $\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}$ stand for the natural, integer, rational, real and complex numbers respectively;
  $\mathbf{Q}_p$ denotes the p-adic numbers and $\mathbf{Z}_p$ the p-adic integers, $\mathbf{A}$ stands

for the adeles of $\mathbf{Q}$ and $\mathbf{A}_f := \prod'_{p<\infty} \mathbf{Q}_p$ the finite adeles;
for a ring R we use the superscript $R^\times$ to denote the invertible elements in R;

- F is a non-archimedean local field of characteristic zero, $\mathfrak{o}$ its ring of integers, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$, ϖ a generator of $\mathfrak{p}$, and q the cardinality of the residue field $\mathfrak{o}/\varpi\mathfrak{o}$; for our global application we will only need $F = \mathbf{Q}_p$;

- M_n denotes the set of $n \times n$ matrices, whose identity element is 1_n;
we use the superscript M_n^{sym} for symmetric matrices, and M_n^+ for the matrices with positive determinant;
we distinguish a set

$$\mathcal{P}_n := \{T \in \frac{1}{2} M_n^{\mathrm{sym}}(\mathbf{Z}) : T \text{ half-integral and positive definite}\},$$

where *half-integral* means that T has integers on the diagonal;
${}^tT$ is the transpose of T and $\operatorname{tr} T$ the trace of T,

$$\operatorname{cont}\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} := \gcd(a,b,c)\,, \quad \operatorname{disc}\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} := b^2 - 4ac$$

are the content and the discriminant of the matrix $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$;

- The letter G will always stand for the group GSp_4 defined as follows:

$$\mathrm{GSp}_4(\mathbf{Q}) := \{g \in \mathrm{GL}_4(\mathbf{Q})\colon\ {}^tg \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix} g = \mu(g) \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}\},$$

where $\mu(g) \in \mathbf{Q}^\times$,

$$\mathrm{Sp}_4(\mathbf{Q}) := \{g \in G(\mathbf{Q})\colon \mu(g) = 1\};$$

We also define the following local subgroups:
the Iwahori subgroup:

$$I := I(1) := I(1,1) = G(\mathfrak{o}) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{pmatrix},$$

the Borel subgroup of level $\mathfrak{p}^n$:

$$I(n) := G(\mathfrak{o}) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} \end{pmatrix},$$

for $n_1 \leq n_2$:

$$I(n_1, n_2) := G(\mathfrak{o}) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{p}^{n_1} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^{n_2} & \mathfrak{p}^{n_2} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^{n_2} & \mathfrak{p}^{n_2} & \mathfrak{p}^{n_1} & \mathfrak{o} \end{pmatrix},$$

the Siegel congruence subgroup of level $\mathfrak{p}$:

$$P_1 := I(0,1) = G(\mathfrak{o}) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \end{pmatrix};$$

the paramodular subgroup of level $\mathfrak{p}$:

$$P_{02} := G(F) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o}/\mathfrak{p} \\ \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{pmatrix};$$

- For $N = \prod_p p^{n_p}$ we define

$$I_N := \prod_{p<\infty} I(n_p),$$

and similarly for $N_1 | N_2$,

$$I_{N_1,N_2} := \prod_{p<\infty} I(n_{1,p}, n_{2,p});$$

$$K_N^* := \prod_{p<\infty} \{g \in \mathrm{GL}_2(\mathbf{Z}_p) \colon g = \begin{pmatrix} * & \\ & * \end{pmatrix} \bmod p^{n_p}\},$$

$$K_N^0 = \prod_{p<\infty} \{g \in \mathrm{GL}_2(\mathbf{Z}_p) \colon g = \begin{pmatrix} * & \\ * & * \end{pmatrix} \bmod p^{n_p}\};$$

- For $N_1|N_2$ and $N \in \mathbf{N}$ we define

$$\Gamma_0(N_1,N_2) := G(\mathbf{Q}) \cap G(\mathbf{R})^+ I_{N_1,N_2} = \mathrm{Sp}_4(\mathbf{Z}) \cap \begin{pmatrix} \mathbf{Z} & N_1\mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ N_2\mathbf{Z} & N_2\mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ N_2\mathbf{Z} & N_2\mathbf{Z} & N_1\mathbf{Z} & \mathbf{Z} \end{pmatrix},$$

the Siegel congruence subgroup of level N:

$$\Gamma_0^{(2)}(N) := \Gamma_0(1,N)$$

and

$$\Gamma_0^{(n)}(N) := \mathrm{Sp}_{2n}(\mathbf{Z}) \cap \{ \begin{pmatrix} A_n & B_n \\ NC_n & D_n \end{pmatrix} : A_n, B_n, C_n, D_n \in M_n(\mathbf{Z})\},$$

and the paramodular subgroup of $\mathrm{Sp}_4(\mathbf{Q})$ of level N:

$$\Gamma^{\mathrm{para}}(N) := \mathrm{Sp}_4(\mathbf{Q}) \cap \begin{pmatrix} \mathbf{Z} & N\mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \mathbf{Z}/N \\ \mathbf{Z} & N\mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ N\mathbf{Z} & N\mathbf{Z} & N\mathbf{Z} & \mathbf{Z} \end{pmatrix}.$$

- For $N \in \mathbf{N}$,

$$\Gamma_0(N) := \mathrm{SL}_2(\mathbf{Z}) \cap \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ N\mathbf{Z} & \mathbf{Z} \end{pmatrix},$$

$$\Gamma^0(N) := \mathrm{SL}_2(\mathbf{Z}) \cap \begin{pmatrix} \mathbf{Z} & N\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{pmatrix};$$

and locally at the place p:

$$\Gamma^0(p^n\mathbf{Z}_p) := \mathrm{SL}_2(\mathbf{Z}_p) \cap \begin{pmatrix} \mathbf{Z}_p & p^n\mathbf{Z}_p \\ \mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix};$$

- For $N \in \mathbf{N}, X \in \mathbf{Z}$ and $\mathcal{S} \subseteq \{p : p|N\}$, we put

$$X_{\mathcal{S}} := \prod_{p \in \mathcal{S}} p^{\mathrm{ord}_p X}, \qquad \text{where} \quad ord_p X := \max\{n \in \mathbf{Z} \colon p^n | X\};$$

$$(X, N^\infty) := X_{\mathcal{S}} \qquad \text{with} \quad \mathcal{S} = \{p : p|N\},$$

and N^∞ denotes a formal number such that $N^l | N^\infty$ for all $l \in \mathbf{Z}^+$;

$$p^k \parallel N \qquad \text{means} \qquad k = ord_p N;$$

$\left(\frac{X}{p}\right)$ denotes the Legendre symbol;

- For a congruence subgroup $\Gamma^{(n)}$ of $\mathrm{Sp}_{2n}(\mathbf{Z})$, we put

$$M_k^{(n)}(\Gamma^{(n)}) := \{\text{Siegel modular forms of weight } k, \text{ level } \Gamma^{(n)}\},$$
$$S_k^{(n)}(\Gamma^{(n)}) := \{\text{cuspidal Siegel modular forms of weight } k, \text{ level } \Gamma^{(n)}\}.$$

Chapter 2

Local theory

2.1 Local Bessel models for GSp_4

We recall the definition of the Bessel model following the exposition of Furusawa [14] and Pitale, Schmidt [38]. Let $S \in M_2(F)$ be a symmetric matrix such that $d = \operatorname{disc} S = -4 \det S \neq 0$. For

$$S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

we define the element

$$\xi = \xi_S = \begin{pmatrix} b/2 & c \\ -a & -b/2 \end{pmatrix}$$

and denote by $F(\xi)$ a two-dimensional F-algebra generated by 1_2 and ξ. Note that

$$\xi^2 = \begin{pmatrix} \frac{d}{4} & \\ & \frac{d}{4} \end{pmatrix}.$$

Depending whether d is a square in $F^\times$ or not, $F(\xi)$ is isomorphic either to $L = F \oplus F$ or to the field $L = F(\sqrt{d})$ via

$$x1_2 + y\xi \longmapsto \begin{cases} x + y\frac{\sqrt{d}}{2} & d \notin (F^\times)^2 \\ (x + y\frac{\sqrt{d}}{2}, x - y\frac{\sqrt{d}}{2}) & d \in (F^\times)^2 \end{cases}. \tag{2.1}$$

The determinant map on $F(\xi)$ corresponds to the norm map on L, defined by $N_{L/F}(z) = z\bar{z}$, where $z \mapsto \bar{z}$ is the usual involution on L fixing F. We define the Legendre symbol as

$$\left(\frac{L}{\mathfrak{p}}\right) = \begin{cases} -1 & \text{if } L/F \text{ is an unramified field extension} \\ 0 & \text{if } L/F \text{ is a ramified field extension} \\ 1 & \text{if } L = F \oplus F \end{cases} .$$

If L is a field, denote by $\mathfrak{o}_L, \mathfrak{p}_L, \varpi_L$ the ring of integers, the maximal ideal of $\mathfrak{o}_L$ and a fixed choice uniformizer in $\mathfrak{o}_L$, correspondingly. If $L = F \oplus F$, let $\mathfrak{o}_L = \mathfrak{o} \oplus \mathfrak{o}$, $\varpi_L = (\varpi, 1)$. Define an ideal $\mathfrak{P} := \mathfrak{p}\mathfrak{o}_L$ in $\mathfrak{o}_L$; note that $\mathfrak{P} = \mathfrak{p}_L$ is prime only if $\left(\frac{L}{\mathfrak{p}}\right) = -1$, otherwise $\mathfrak{P} = \mathfrak{p}_L^2$ if $\left(\frac{L}{\mathfrak{p}}\right) = 0$ and $\mathfrak{P} = \mathfrak{p} \oplus \mathfrak{p}$ if $\left(\frac{L}{\mathfrak{p}}\right) = 1$.

We define a subgroup $T = T_S$ of GL_2 by

$$T(F) = \{g \in \mathrm{GL}_2(F) \mid {}^t g S g = \det(g) S\} . \tag{2.2}$$

It is not hard to verify that $T(F) = F(\xi)^\times$, so that $T(F) \cong L^\times$. We identify $T(F)$ with $L^\times$ via (2.1). We can consider T as a subgroup of G[1] via

$$T \ni g \longmapsto \begin{pmatrix} g & \\ & \det g \cdot {}^t g^{-1} \end{pmatrix} \in G.$$

Let us denote by U the subgroup of G defined by

$$U = \{u(X) = \begin{pmatrix} 1_2 & X \\ & 1_2 \end{pmatrix} \mid {}^t X = X\},$$

and finally let R be the subgroup of G defined by $R = TU$.

We fix a non-trivial additive character ψ of F such that ψ is trivial on $\mathfrak{o}$, but non-trivial on $\mathfrak{p}^{-1}$, and define the character $\theta = \theta_S$ on $U(F)$ by

$$\theta(u(X)) := \psi(\mathrm{tr}\,(SX)) . \tag{2.3}$$

Let Λ be a character of $T(F)$ such that $\Lambda|_{F^\times} = 1$. Denote by $\Lambda \otimes \theta$ the character of $R(F)$ defined by $(\Lambda \otimes \theta)(tu) = \Lambda(t)\theta(u)$ for $t \in T(F), u \in U(F)$.

Let π be an irreducible admissible representation of the group $G(F)$ with trivial central character. We say that such[2] a π has a *local Bessel model* of type (Λ, θ) if π is isomorphic to a subrepresentation of the space

[1] Recall that we denote by G the group GSp_4.

[2] We define a local Bessel model only for representations with trivial central character, because our main results are formulated for such representations. To define a local Bessel model of type (Λ, θ) for a representation π with the central character ω_π, one should assume that $\Lambda|_{F^\times} = \omega_\pi$ and leave the rest unchanged.

of all locally constant functions B on $G(F)$ satisfying the local Bessel transformation property

$$B(rg) = (\Lambda \otimes \theta)(r)B(g) \text{ for all } r \in R(F) \text{ and } g \in G(F)\,.$$

It is known by [31], [42] that if a local Bessel model exists, then it is unique. If the local Bessel model for π exists, we denote it by $\mathcal{B}^{\pi}_{\Lambda,\theta}$. In this case, we fix a (unique up to scalar) isomorphism of representations $\pi \to \mathcal{B}^{\pi}_{\Lambda,\theta}$ and denote the image of any $\phi \in \pi$ by B_ϕ.

In the Lemma below we explain how to switch between Bessel models defined with respect to different matrices S. Together with Lemma 1.1, [39] that will allow us to assume, without any loss of generality, that the entries a, b, c and the discriminant $d = b^2 - 4ac$ of S satisfy the following conditions:

- $a, b \in \mathfrak{o}$ and $c \in \mathfrak{o}^\times$.
- If $d \notin (F^\times)^2,$ then d is a generator of the discriminant of L/F. (2.4)
- If $d \in (F^\times)^2,$ then $d \in \mathfrak{o}^\times$.

Lemma 2.1.1. *Let $S \in M_2(F)$ be symmetric, and let Λ be a character of the associated group $T_S(F)$. Let $A \in \mathrm{GL}_2(F)$ and $\alpha \in F^\times$. Let $S' = \alpha\, {}^tASA$. Then $T_{S'}(F) = A^{-1}T_S(F)A$, so that*

$$\Lambda'(t') = \Lambda(At'A^{-1}), \qquad t' \in T_{S'}(F)\,,$$

defines a character of $T_{S'}(F)$. Let π be an irreducible admissible representation of $G(F)$. Then π has a local Bessel model of type (Λ, θ_S) if and only if it has a local Bessel model of type $(\Lambda', \theta_{S'})$.

Proof. Indeed, if $B \in \mathcal{B}^{\pi}_{\Lambda,\theta}$, then $B'(g) := B(\left(\begin{smallmatrix} A & \\ & \alpha^{-1}\,{}^tA^{-1}\end{smallmatrix}\right) g),\ g \in G(F)$ satisfies the Bessel transformation property and the map $B \to B'$ gives rise to a local Bessel model $\mathcal{B}^{\pi}_{\Lambda',\theta_{S'}}$. $\square$

2.2 Sugano's formula

We now investigate more closely the case when π is *spherical*, that is, π has a non-zero $G(\mathfrak{o})$-invariant vector. Such a representation is a constituent of a representation parabolically induced from an unramified character γ of the Borel subgroup of $G(F)$. The values of the character γ at the matrices

$$\begin{pmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{pmatrix}, \begin{pmatrix} \varpi &&& \\ & 1 && \\ && 1 & \\ &&& \varpi \end{pmatrix}, \begin{pmatrix} 1 &&& \\ & 1 && \\ && \varpi & \\ &&& \varpi \end{pmatrix}, \begin{pmatrix} 1 &&& \\ & \varpi && \\ && \varpi & \\ &&& 1 \end{pmatrix}$$

are called the *Satake parameters* of π and determine the isomorphism class of π. Because central character of π is trivial, we can call them in turn $\alpha, \beta, \alpha^{-1}, \beta^{-1}$.

Throughout this section we assume the following:

(i) π is a spherical representation of $G(F)$,

(ii) $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ with a, b, c satisfying the conditions (2.4),

(iii) $\theta = \theta_S$ is the character of $U(F)$ as in (2.3),

(iv) Λ is a character of $T(F)$ that is invariant under the subgroup

$$T(n) := T(F) \cap \left\{ g \in \mathrm{GL}_2(\mathfrak{o}) : g = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \bmod \mathfrak{p}^n,\ \lambda \in \mathfrak{o}^\times \right\},$$

for some non-negative integer n.

The next Lemma shows equivalent ways of writing the group $T(n)$. Thanks to this, our definition coincides with the one used in [35] and [38].

Lemma 2.2.1. *The group $T(n)$ defined above is isomorphic to each of the following:*

$$T(F) \cap \{ g \in \mathrm{GL}_2(\mathfrak{o}) : g = \begin{pmatrix} * & \\ * & * \end{pmatrix} \bmod \mathfrak{p}^n \},$$

$$T(F) \cap \{ g \in \mathrm{GL}_2(\mathfrak{o}) : g = \begin{pmatrix} * & \\ & * \end{pmatrix} \bmod \mathfrak{p}^n \}$$

and (under the isomorphism $T(F) \cong L^\times$)

$$\mathfrak{o}^\times (1 + \mathfrak{P}^n) \cap \mathfrak{o}_L^\times .$$

Moreover, every character of $T(F)$ that is trivial on $\mathfrak{o}^\times$ is trivial on $T(n)$ for n big enough.

Proof. By Lemma 3.1.1, [40], $\mathfrak{o}_L = \mathfrak{o} + \mathfrak{o}\xi_0$, where

$$\xi_0 = \begin{cases} \frac{-b+\sqrt{d}}{2} & \text{if } L \text{ is a field} \\ \left(\frac{-b+\sqrt{d}}{2}, \frac{-b-\sqrt{d}}{2} \right) & \text{if } L = F \oplus F \end{cases}. \tag{2.5}$$

Therefore by the identification (2.1),

$$\mathfrak{o}_L = \{\begin{pmatrix} x & yc \\ -ya & x - yb \end{pmatrix} : x, y \in \mathfrak{o}\}.$$

Hence, under the assumptions (2.4) and via the isomorphism $T(F) \cong L^\times$, the group $T(\mathfrak{o}) := T(F) \cap \mathrm{GL}_2(\mathfrak{o})$ is isomorphic to $\mathfrak{o}_L^\times$ and $T(F) \cap M_2(\mathfrak{o}) \cong \mathfrak{o}_L$. In this way

$$\lambda(1 + \mathfrak{P}^n) \cap \mathfrak{o}_L^\times = \lambda(1_2 + \mathfrak{p}^n(T(F) \cap M_2(\mathfrak{o}))) \cap T(\mathfrak{o})$$

and thus

$$T(n) \cong \mathfrak{o}^\times\,(1 + \mathfrak{P}^n) \cap \mathfrak{o}_L^\times\,. \tag{$\star$}$$

Assume now that $g \in T(F) \cap \mathrm{GL}_2(\mathfrak{o})$ is congruent to $\left(\begin{smallmatrix} * & \\ * & * \end{smallmatrix}\right)$ mod $\mathfrak{p}^n$. We already know that g must be of the form $x1_2 + y\left(\begin{smallmatrix} & c \\ -a & -b \end{smallmatrix}\right)$ with $x, y \in \mathfrak{o}$. However, because $c \in \mathfrak{o}^\times$, we have $y \in \mathfrak{p}^n\mathfrak{o}$, and thus $g = \left(\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right)$ mod $\mathfrak{p}^n$, which means that $g \in T(n)$. The other inclusions are clear.

To prove the last assertion we use the isomorphism ($\star$). Because $\mathfrak{o}^\times$ is compact and $\{\lambda(1 + \mathfrak{P}^n) \cap \mathfrak{o}_L^\times : n \in \mathbf{N}\}$ gives a set of neighbourhoods of each $\lambda \in \mathfrak{o}^\times$ in $\mathfrak{o}_L^\times$, so if Λ is trivial on $\mathfrak{o}^\times$, it must be trivial on $\{\lambda(1 + \mathfrak{P}^n) \cap \mathfrak{o}_L^\times : \lambda \in \mathfrak{o}^\times\}$ for n big enough. $\square$

Definition 2.2.1. The smallest integer n for which Λ is $T(n)$-invariant or, equivalently,

$$\min\{n \geq 0 : \Lambda|_{(1+\mathfrak{P}^n)\cap\mathfrak{o}_L^\times} = 1\}$$

will be denoted by $c(\Lambda)$.

Under the assumptions (i)-(iv), π has a local Bessel model of type (Λ, θ) for Λ as specified in Table 2. For example, if π is an irreducible spherical principal series representation (type I), such a local Bessel model exists for *all* Λ. This model contains a unique (up to multiples) $G(\mathfrak{o})$-invariant vector, which we will denote by $B_\pi^{(0)}(\Lambda, \theta)$ or by $B_\pi^{(0)}$. The following results are due to Sugano [60].

Theorem 2.2.1 (Sugano; [60])**.** *Assume (i)-(iv), let Λ be such that the local Bessel model $\mathcal{B}^\pi_{\Lambda,\theta}$ exists and put*

$$h(l, m) := \begin{pmatrix} \varpi^{l+2m} & & & \\ & \varpi^{l+m} & & \\ & & 1 & \\ & & & \varpi^m \end{pmatrix} \tag{2.6}$$

for $l, m \in \mathbf{Z}$, $m \geq 0$. *Then*

1. $B_\pi^{(0)}(h(l,m)) = 0$ *if* $l < 0$ *or* $m < c(\Lambda)$.

2. $B_\pi^{(0)}(h(0,c(\Lambda))) \neq 0$.

Because of the above theorem, if the local Bessel model $\mathcal{B}^\pi_{\Lambda,\theta}$ exists, we can and will henceforth normalize $B_\pi^{(0)}$ so that $B_\pi^{(0)}(h(0,c(\Lambda))) = 1$. For brevity, we shall henceforth denote

$$U_\pi(l,m) := q^{2m+3l/2} B_\pi^{(0)}(h(l,m+c(\Lambda)))\,.$$

Then Sugano's formula states that

Theorem 2.2.2 (Sugano; [60]). *Let* π *be a spherical representation with Satake parameters* $\alpha, \beta, \alpha^{-1}, \beta^{-1}$. *Assume that* π *admits a local Bessel model and let* $U_\pi(l,m)$ *be as above. Then the generating function*

$$C(X,Y) = C(X,Y;\alpha,\beta) = \sum_{l\geq 0}\sum_{m\geq 0} U_\pi(l,m) X^m Y^l \tag{2.7}$$

is a rational function given by

$$C(X,Y) = \frac{H(X,Y)}{P(X)Q(Y)},$$

where

$$\begin{aligned}
P(X) &= (1-\alpha\beta X)(1-\alpha\beta^{-1}X)(1-\alpha^{-1}\beta X)(1-\alpha^{-1}\beta^{-1}X),\\
Q(Y) &= (1-\alpha Y)(1-\beta Y)(1-\alpha^{-1}Y)(1-\beta^{-1}Y),\\
H(X,Y) &= (1+XY^2)\left(M_1(X)(1+X) + q^{-1/2}\epsilon\sigma(\alpha,\beta)X^2\right)\\
&\quad - XY\left(\sigma(\alpha,\beta)M_1(X) - q^{-1/2}\epsilon M_2(X)\right) - q^{-1/2}\epsilon P(X)Y\\
&\quad + q^{-1}\left(\frac{\Lambda}{\mathfrak{p}}\right)P(X)Y^2,
\end{aligned}$$

in terms of auxiliary polynomials given by

$$\begin{aligned}
&\sigma(\alpha,\beta) = \alpha+\beta+\alpha^{-1}+\beta^{-1},\ \tau(\alpha,\beta) = 1+\alpha\beta+\alpha\beta^{-1}+\alpha^{-1}\beta+\alpha^{-1}\beta^{-1},\\
&M_1(X) = 1 - \left(q - \left(\frac{\Lambda}{\mathfrak{p}}\right)\right)^{-1}\left(q^{1/2}\epsilon\sigma(\alpha,\beta) - \left(\frac{\Lambda}{\mathfrak{p}}\right)(\tau(\alpha,\beta)-1) - \epsilon^2\right)X\\
&\qquad - q^{-1}\left(\frac{\Lambda}{\mathfrak{p}}\right)X^2,\\
&M_2(X) = 1 - \tau(\alpha,\beta)X - \tau(\alpha,\beta)X^2 + X^3,
\end{aligned}$$

where

$$\left(\frac{\Lambda}{\mathfrak{p}}\right) = \begin{cases} \left(\frac{L}{\mathfrak{p}}\right) & \textit{if } c(\Lambda) = 0 \\ 0 & \textit{if } c(\Lambda) > 0 \end{cases},$$

$$\epsilon = \begin{cases} 0 & \textit{if } \left(\frac{L}{\mathfrak{p}}\right) = -1 \textit{ or } c(\Lambda) > 0 \\ \Lambda(\varpi_L) & \textit{if } \left(\frac{L}{\mathfrak{p}}\right) = 0, c(\Lambda) = 0 \\ \Lambda(\varpi_L) + \Lambda(\varpi\varpi_L^{-1}) & \textit{if } \left(\frac{L}{\mathfrak{p}}\right) = 1, c(\Lambda) = 0 \end{cases}.$$

2.3 Bessel models for non-spherical representations

In general, given characters Λ and θ as defined in Section 2.1, a non-spherical (irreducible admissible) representation π may or may not have a (Λ, θ)-Bessel model. In case π is non-supercuspidal, Roberts and Schmidt [46] used a classification of such representations due to Sally and Tadić [53] and for all of them provided a complete list of characters Λ for which the (Λ, θ)-Bessel model exists. We summarise a part of their result in a Table 2 below for the representations that will be of special interest to us (mainly because of the knowledge on their test vectors - cf. Theorems 2.3.1, 2.3.2, 2.3.3).

Throughout this section we assume that the characters Λ, θ satisfy the conditions (ii)-(iv). We assume also that π is an irreducible admissible representation with trivial central character.

Definition 2.3.1. Let $\phi \in V_\pi$, and the characters Λ, θ be such that $\mathcal{B}^{\pi}_{\Lambda,\theta}$ exists. Let $h(l, m)$ be as in (2.6) and define

$$m_{\phi,\Lambda} := \min(\{m : B_\phi(h(0, m)) \neq 0\} \cup \{\infty\}). \tag{2.8}$$

Whenever $m_{\phi,\Lambda} < \infty$, we normalise $B_\phi(h(0, m_{\phi,\Lambda}))$ so that it is equal to 1.

Note that $m_{\phi,\Lambda} = c(\Lambda)$ if π is spherical and a (Λ, θ)-Bessel model for π exists. In this section we will prove that this continues to hold for some other cases.

We will now recall some results due to Pitale and Schmidt [34], [38] on local Bessel models for non-spherical representations that have a non-zero vector fixed under the subgroup I or P_1. These theorems identify test vectors for the aforementioned representations; the representations are classified according to Table 1 (taken from [38]). Table 2 (taken from [46]) provides precise conditions that character Λ needs to satisfy so that a given representation has a (Λ, θ)-Bessel model.

Table 1. The Iwahori-spherical representations of $\mathrm{GSp}_4(F)$ and the dimensions of their spaces of fixed vectors under the parahoric subgroups. The symbol ν stands for the absolute value on $F^\times$, normalized such that $\nu(\varpi) = q^{-1}$, σ and ξ denote the non-trivial, quadratic characters of $F^\times$, ξ unramified.

Type		representation	$\mathrm{GSp}_4(\mathfrak{o})$	P_{02}	P_1	I
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	1	2	4	8
II	a	$\chi \mathrm{St}_{\mathrm{GL}_2} \rtimes \sigma$	0	1	1	4
	b	$\chi 1_{\mathrm{GL}_2} \rtimes \sigma$	1	1	3	4
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}_2}$	0	0	2	4
	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}_2}$	1	2	2	4
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}_4}$	0	0	0	1
	b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}_2})$	0	0	2	3
	c	$L(\nu^{3/2} \mathrm{St}_{\mathrm{GL}_2}, \nu^{-3/2}\sigma)$	0	1	1	3
	d	$\sigma 1_{\mathrm{GSp}_4}$	1	1	1	1
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	0	0	0	2
	b	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}_2}, \nu^{-1/2}\sigma)$	0	1	1	2
	c	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}_2}, \xi\nu^{-1/2}\sigma)$	0	1	1	2
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	1	0	2	2
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	0	0	1	3
	b	$\tau(T, \nu^{-1/2}\sigma)$	0	0	1	1
	c	$L(\nu^{1/2} \mathrm{St}_{\mathrm{GL}_2}, \nu^{-1/2}\sigma)$	0	1	0	1
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	1	1	2	3

Table 2. The Bessel models of the irreducible, admissible representations of $\mathrm{GSp}_4(F)$ that can be obtained via induction from the Borel subgroup. The column $L \leftrightarrow \xi$ indicates that the field L is the quadratic extension of F corresponding to the non-trivial, quadratic character ξ of $F^\times$. The pairs of characters (χ_1, χ_2) in the $L = F \oplus F$ column for types IIIb and IVc refer to the characters of $T = \{\mathrm{diag}(a, b, b, a) : a, b \in F^\times\}$ given by $\mathrm{diag}(a, b, b, a) \mapsto \chi_1(a)\chi_2(b)$.

<table>
<tr><th colspan="2" rowspan="3">Type</th><th colspan="3">(Λ, θ)-Bessel functional exists exactly for ...</th></tr>
<tr><th rowspan="2">$L = F \oplus F$</th><th colspan="2">L/F a field extension</th></tr>
<tr><th>$L \leftrightarrow \xi$</th><th>$L \not\leftrightarrow \xi$</th></tr>
<tr><td colspan="2">I</td><td>all Λ</td><td colspan="2">all Λ</td></tr>
<tr><td>II</td><td>a</td><td>all Λ</td><td colspan="2">$\Lambda \neq (\chi\sigma) \circ \mathrm{N}_{L/F}$</td></tr>
<tr><td></td><td>b</td><td>$\Lambda = (\chi\sigma) \circ \mathrm{N}_{L/F}$</td><td colspan="2">$\Lambda = (\chi\sigma) \circ \mathrm{N}_{L/F}$</td></tr>
<tr><td>III</td><td>a</td><td>all Λ</td><td colspan="2">all Λ</td></tr>
<tr><td></td><td>b</td><td>$\Lambda \in \{(\chi\sigma, \sigma), (\sigma, \chi\sigma)\}$</td><td colspan="2">—</td></tr>
<tr><td>IV</td><td>a</td><td>all Λ</td><td colspan="2">$\Lambda \neq \sigma \circ \mathrm{N}_{L/F}$</td></tr>
<tr><td></td><td>b</td><td>$\Lambda = \sigma \circ \mathrm{N}_{L/F}$</td><td colspan="2">$\Lambda = \sigma \circ \mathrm{N}_{L/F}$</td></tr>
<tr><td></td><td>c</td><td>$\Lambda = (\nu^{\pm 1}\sigma, \nu^{\mp 1}\sigma)$</td><td colspan="2">—</td></tr>
<tr><td></td><td>d</td><td>—</td><td colspan="2">—</td></tr>
<tr><td>V</td><td>a</td><td>all Λ</td><td>$\Lambda \neq \sigma \circ \mathrm{N}_{L/F}$</td><td>$\sigma \circ \mathrm{N}_{L/F} \neq \Lambda \neq (\xi\sigma) \circ \mathrm{N}_{L/F}$</td></tr>
<tr><td></td><td>b</td><td>$\Lambda = \sigma \circ \mathrm{N}_{L/F}$</td><td>—</td><td>$\Lambda = \sigma \circ \mathrm{N}_{L/F}$</td></tr>
<tr><td></td><td>c</td><td>$\Lambda = (\xi\sigma) \circ \mathrm{N}_{L/F}$</td><td>—</td><td>$\Lambda = (\xi\sigma) \circ \mathrm{N}_{L/F}$</td></tr>
<tr><td></td><td>d</td><td>—</td><td>$\Lambda = \sigma \circ \mathrm{N}_{L/F}$</td><td>—</td></tr>
<tr><td>VI</td><td>a</td><td>all Λ</td><td colspan="2">$\Lambda \neq \sigma \circ \mathrm{N}_{L/F}$</td></tr>
<tr><td></td><td>b</td><td>—</td><td colspan="2">$\Lambda = \sigma \circ \mathrm{N}_{L/F}$</td></tr>
<tr><td></td><td>c</td><td>$\Lambda = \sigma \circ \mathrm{N}_{L/F}$</td><td colspan="2">—</td></tr>
<tr><td></td><td>d</td><td>$\Lambda = \sigma \circ \mathrm{N}_{L/F}$</td><td colspan="2">—</td></tr>
</table>

Theorem 2.3.1 (Pitale; [34]). *Assume (ii)-(iv). Let π be of type IVa. If π has a (Λ,θ)-Bessel model and $\phi \in V_\pi$ is a vector fixed by I, then:*

$$m_{\phi,\Lambda} = \begin{cases} 0 & \text{if } c(\Lambda) = 0 \\ 0 & \text{if } c(\Lambda) = 1 \text{ and either } \left(\frac{L}{\mathfrak{p}}\right) \neq 1 \text{ or } \Lambda(1,\varpi) \neq \sigma(\varpi) \\ c(\Lambda) - 1 & \text{if } c(\Lambda) > 1 \end{cases}$$

(σ is a character of $F^\times$ as in Table 1).

Theorem 2.3.2 (Pitale, Schmidt; [38]). *Assume (ii)-(iv). Let π be a representation of $G(F)$ that is not spherical but has a one-dimensional space of P_1-invariant vectors (i.e. π is of type IIa, IVc, Vb, Vc, VIa or VIb). Assume that π admits a (Λ,θ)-Bessel model and let ϕ be an element in this model spanning the space of P_1-invariant vectors. Then*

$$m_{\phi,\Lambda} = c(\Lambda)$$

unless π is of type IIa, $\left(\frac{L}{\mathfrak{p}}\right) = 1$ and $\Lambda(1,\varpi) = -\omega = \Lambda(\varpi,1)$ (ω denotes an eigenvalue of the Atkin-Lehner operator at ϕ).

Theorem 2.3.3 (Pitale, Schmidt; [38]). *Assume (ii)-(iv). Let π be a representation of $G(F)$ that is not spherical but has a two-dimensional space of P_1-invariant vectors (i.e. π is of type IIIa or IVb). Assume that π admits a (Λ,θ)-Bessel model. Then the space of P_1-invariant vectors is spanned by common eigenvectors for the Hecke operators*

$$T_{1,0}(v) := \frac{1}{\mathrm{vol}(\mathrm{P}_1)} \int_{P_1 h(1,0) P_1} \pi(g) v \, dg$$

and

$$T_{0,1}(v) := \frac{1}{\mathrm{vol}(\mathrm{P}_1)} \int_{P_1 h(0,1) P_1} \pi(g) v \, dg \,,$$

and if ϕ is any such eigenvector, then

$$m_{\phi,\Lambda} = c(\Lambda) \,.$$

Moreover, if ϕ satisfies the assumptions of Theorem 2.3.2 or 2.3.3, $T_{1,0}\phi = \lambda\phi$ and $T_{0,1}\phi = \mu\phi$, then ([38], Proposition 6.1)

$$B_\phi(h(l+1,m)) = \lambda q^{-3} B_\phi(h(l,m)) \quad \text{for all } l,m \geq 0 \,,$$

and for $l \geq 0, m \geq c(\Lambda)$

$$q^4 B_\phi(h(l,m+2)) - \mu B_\phi(h(l,m+1)) + \lambda^2 q^{-3} \Lambda(\varpi) B_\phi(h(l,m)) = 0$$

or, more generally, for $l \geq 0$,

$$Y^{-c(\Lambda)} \sum_{m=c(\Lambda)}^{\infty} B_\phi(h(l,m)) Y^m = \frac{1 - \kappa q^{-4} Y}{1 - \mu q^{-4} Y + \lambda^2 q^{-7} \Lambda(\varpi) Y^2} B_\phi(h(l, c(\Lambda)))\,,$$

where κ is an (explicit) constant depending on eigenvalues λ, μ, character Λ, F and L, and such that $\kappa = 0$ if $c(\Lambda) > 0$ (values of λ and μ are listed in Table 4, [38] in terms of invariants of π).

Remark. The upshot of this section is that for almost[3] every representation π listed in Table 2 that admits a (Λ, θ)-Bessel model, and for each new vector $\phi \in V_\pi$ fixed under the action of $G(\mathfrak{o}), P_1$ or I, we know the (finite) value of $m_{\phi,\Lambda}$. In most cases it is equal to $c(\Lambda)$.

Unfortunately, we cannot use these theorems for vectors ϕ that are invariant under the action of $I(n_1, n_2)$ with $n_2 > 1$. However, by an easy adaptation of the proof given in [34], we provide a lower bound for $m_{\phi,\Lambda}$:

Lemma 2.3.1. *Assume (ii)-(iv). Let π be a representation of $G(F)$ that has an $I(n_1,n_2)$-invariant vector ϕ for some $n_2 \geq n_1 \geq 0$. Then*

$$m_{\phi,\Lambda} \geq \max(0, c(\Lambda) - n_1)\,.$$

Proof. The proof is basically the same as the one for Lemma 3.5 in [34], but we write it down for the sake of completeness.

It is clear that $m_{\phi,\Lambda} \geq 0$. We may assume then that $c(\Lambda) > n_1$. Let $m < c(\Lambda) - n_1$, and $l \in \mathbf{Z}$. It is enough to find an element $k \in I(n_1, n_2)$ such that $h(l,m)k = th(l,m)$, $t \in T(m + n_1)$ and $\Lambda(t) \neq 1$, because then

$$B_\phi(h(l,m)) = B_\phi(h(l,m)k) = B_\phi(th(l,m)) = \Lambda(t) B_\phi(h(l,m)) = 0\,.$$

An easy calculation shows that we may take

$$k = \begin{pmatrix} 1+x & cy\varpi^{-m} & & \\ -ay\varpi^m & 1+x-by & & \\ & & 1+x-by & ay\varpi^m \\ & & -cy\varpi^{-m} & 1+x \end{pmatrix},$$

[3] With an exception of type Va and VIc, and with some mild assumptions on types IIa, IVa.

where $x, y \in \mathfrak{p}^{m+n_1}$ are such that $\Lambda(1 + x + y\xi_0) \neq 1$ and ξ_0 as in (2.5) (recall that $1 + x + y\xi_0 \in \mathfrak{o}^\times(1 + \mathfrak{P}^{m+n_1}) \cap \mathfrak{o}_L^\times \cong T(m + n_1)$ by Lemma 2.2.1). □

The aforementioned theorems show that this bound is optimal in many cases. Therefore it makes sense to make the following definition.

Definition 2.3.2. Let π be a representation of $G(F)$ that has an $I(n_1, n_2)$-invariant vector ϕ for some $n_2 \geq n_1 \geq 0$. We say that ϕ is optimal if for all characters Λ of $T(F)$ one of the following holds:

- π has no (Λ, θ)-Bessel model,
- π has a (Λ, θ)-Bessel model and $m_{\phi,\Lambda} = \max(0, c(\Lambda) - n_1)$.

Remark. The vector $\phi \in V_\pi$ is optimal whenever

(i) $n_1 = n_2 = 0$ (i.e. π is spherical);

(ii) $n_1 = 0$, $n_2 = 1$ and either π is not of type IIa or $\left(\frac{L}{\mathfrak{p}}\right) \neq 1$;

(iii) $n_1 = n_2 = 1$ and π is of type IVa.

Chapter 3

Ray class groups and $\Gamma^0(N)$-equivalence

Throughout this chapter we are interested in the elements of the set

$$\mathcal{P}_2 := \left\{ T \in \frac{1}{2} M_2^{\mathrm{sym}}(\mathbf{Z}) : T \text{ half-integral and positive definite} \right\}, \tag{3.1}$$

where we call a matrix T *half-integral* if it has integers on the diagonal. These matrices may be characterised according to their *content* and *discriminant*, defined as

$$\operatorname{cont}\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} := \gcd(a,b,c)\,, \quad \operatorname{disc}\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} := b^2 - 4ac\,.$$

In particular, each discriminant can be written as dM^2L^2, where d is a *fundamental discriminant*[1] and L is a content of the matrix. From now on d will denote a negative fundamental discriminant.

Definition. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$. We say that two matrices T and T' are Γ-*equivalent* if there exists a matrix $A \in \Gamma$ such that $T' = {}^tATA$.

It is easy to see that this relation preserves discriminant and content of the matrices. Therefore it makes sense to make the following definition:

Definition. Let d be a negative fundamental discriminant, and L, M positive integers. For any congruence subgroup Γ of $\mathrm{SL}_2(\mathbf{Z})$ we define the set

[1]Recall that d is a fundamental discriminant if d is square-free and $d \equiv 1 \pmod 4$ or $d = 4d'$, d' square-free and $d' \equiv 2, 3 \pmod 4$. Or, equivalently, if $d = 1$ or d is the discriminant of a quadratic number field.

of Γ-equivalence classes

$$H(dM^2, L; \Gamma) := \{T \in \mathcal{P}_2 \colon \operatorname{disc} T = dL^2M^2, \operatorname{cont} T = L\}/ \sim_\Gamma ,$$

where

$$T \sim_\Gamma T' \qquad \Longleftrightarrow \qquad \exists_{A \in \Gamma} T' = {}^tATA\,.$$

Because of the relation (5.2) satisfied by Fourier coefficients of Siegel modular forms, we are especially interested in the set $H(dM^2, L; \Gamma^0(N))$, where

$$\Gamma^0(N) := \{g \in \mathrm{SL}_2(\mathbf{Z}) : g \equiv \left(\begin{smallmatrix} * & \\ * & * \end{smallmatrix}\right) \bmod N\}\,.$$

It is well-known that when $M = N = 1$, the set $H(d, L; \Gamma^0(1))$ is isomorphic to the ideal class group of $\mathbf{Q}(\sqrt{d})$. As we shall see, when $M, N > 1$ the situation is more complicated. In [35], Pitale, Saha and Schmidt found a bijection between $H(dM^2, L; \Gamma^0(1))$ and a certain ray class group of $\mathbf{Q}(\sqrt{d})$, which we will call later $\mathrm{Cl}_d(M)$. In the next section we are going to extend their result to $N > 1$.

3.1 Construction of an endomorphism

Fix positive integers M, N and a negative fundamental discriminant d. Let

$$S(d) = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} := \begin{cases} \begin{pmatrix} \frac{-d}{4} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } d \equiv 0 \pmod 4 \\ \begin{pmatrix} \frac{1-d}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} & \text{if } d \equiv 1 \pmod 4 \end{cases} \tag{3.2}$$

and let $T = T_{S(d)}$ be a group defined in section 2.1.

Definition 3.1.1. For $N = \prod_p p^{n_p}$ define

$$T_N := \prod_{p<\infty} T(n_p) \quad \text{and} \quad \mathrm{Cl}_d(N) := T(\mathbf{A})/T(\mathbf{Q})T(\mathbf{R})T_N\,,$$

where $T(n_p) \subseteq T(\mathbf{Q}_p)$ is as in section 2.2 and by $T(0)$ we mean the maximal compact subgroup $T(\mathbf{Z}_p) := T(\mathbf{Q}_p) \cap \mathrm{GL}_2(\mathbf{Z}_p)$ of $T(\mathbf{Q}_p)$.

Because of the isomorphism described in Lemma 2.2.1, we may view $\mathrm{Cl}_d(N)$ as a *ray class group* of $\mathbf{Q}(\sqrt{d})$.

Basing on the argument of [35], we will now describe a certain map from $\mathrm{Cl}_d(N')$ to $H(dM^2, L; \Gamma^0(N))$, where N' is any integer divisible by MN.

Let $c \in \mathrm{Cl}_d(N')$ and let $t_c \in T(\mathbf{A})$ be a representative for c such that $t_c \in \prod_{p<\infty} T(\mathbf{Q}_p)$. By strong approximation we can write $t_c = \gamma_c m_c \kappa_c$, where $\gamma_c \in \mathrm{GL}_2(\mathbf{Q})$, $m_c \in \mathrm{GL}_2(\mathbf{R})^+$ and $\kappa_c \in K^*_{N'}$. Also, denote by $(\gamma_c)_f$ the finite part of γ_c when considered as an element of $\mathrm{GL}_2(\mathbf{A})$, thus we have the equality $(\gamma_c)_f = \gamma_c m_c$, as elements of $\mathrm{GL}_2(\mathbf{A})$. Let

$$S_c := \det(\gamma_c)^{-1}\, {}^t\gamma_c S(d) \gamma_c\,; \tag{3.3}$$

it is a positive definite, half-integral, symmetric matrix of discriminant d and content 1 (cf. [14], p. 209). Put

$$\phi_{L,M}(c) = L \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix} \right) S_c \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix} \right) . \tag{3.4}$$

Then $\phi_{L,M}(c)$ is a matrix of discriminant dM^2L^2 and content L.

Note that the matrices $\phi_{L,M}(c)$ constructed above are not uniquely defined, as they depend on the choice of t_c and κ_c. However, the definition is correct for $\Gamma^0(N)$-equivalence classes.

Proposition 3.1.1. *Assume that $MN|N'$. Then the map $\tilde{\phi}_{L,M} = \tilde{\phi}_{L,M;N'}$ from $\mathrm{Cl}_d(N')$ to $H(dM^2, L; \Gamma^0(N))$, sending c to $\phi_{L,M}(c)$ is well-defined. Moreover, if $N' = MN$, $\tilde{\phi}_{L,M;N'}$ is injective.*

Proof. This follows almost immediately from the proof of Proposition 5.3, [35]. The first part goes without any change. To show injectivity, it suffices to exchange a group $\mathrm{SL}_2(\mathbf{Z})$ occurring in the second part of the proof with $\Gamma^0(N)$. More precisely, if we assume that there exists a matrix $A \in \Gamma^0(N)$ such that ${}^tA\phi_{L,M}(c_2)A = \phi_{L,M}(c_1)$, then A must be, in fact, an element of $\Gamma^0(N) \cap \Gamma_0(M)$. Observe that ${}^tRS_{c_2}R = S_{c_1}$ for $R = \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix} \right) A \left(\begin{smallmatrix} 1/M & \\ & 1 \end{smallmatrix} \right) \in \Gamma^0(MN)$, and so if γ_1, γ_2 correspond to S_{c_1}, S_{c_2} via (3.3), then $\gamma_2 R \gamma_1^{-1} \in T(\mathbf{Q})$. Therefore, if we take $t_1 = \gamma_1\gamma_{1,\infty}^{-1}\kappa_1$ and $t_2 = \gamma_2\gamma_{2,\infty}^{-1}\kappa_2$ as representatives in $\prod_{p<\infty} T(\mathbf{Q}_p)$ of c_1 and c_2, then

$$\gamma_2 R \gamma_1^{-1} \in T(\mathbf{Q}) \cap t_2 \mathrm{GL}_2(\mathbf{R})^+ K^0_{MN} t_1^{-1} .$$

This means that t_1 and t_2 represent the same element in $\mathrm{Cl}_d(N')$, provided $N' = MN$. ∎

3.2 The image of $\tilde{\phi}_{L,M}$

We start with the observation that the image of the map $\tilde{\phi}_{L,M}$ from $\mathrm{Cl}_d(N')$ to $H(dM^2, L; \Gamma^0(N))$ constructed above does not depend on N', but only on MN.

Lemma 3.2.1. *Let $MN|N_1'|N_2'$, and let $\rho\colon \mathrm{Cl}_d(N_2') \to \mathrm{Cl}_d(N_1')$ be the natural projection. Then the following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{Cl}_d(N_2') \arrow[rr, "\rho"] \arrow[dr, "\tilde{\phi}_{L,M;N_2'}"'] & & \mathrm{Cl}_d(N_1') \arrow[dl, "\tilde{\phi}_{L,M;N_1'}"] \\
& H(dM^2, L; \Gamma^0(N)) &
\end{tikzcd}$$

Proof. This follows by construction. Let $c \in \mathrm{Cl}_d(N_1')$ and $c_1, c_2, \ldots, c_t$ be the elements of $\mathrm{Cl}_d(N_2')$ that the map ρ sends to c. Choose distinct $i, j \in \{1, 2, \ldots, t\}$. We will show that $\phi_{L,M}(c_i)$ and $\phi_{L,M}(c_j)$ are $\Gamma^0(N)$-equivalent. For this it suffices to find $\gamma \in \Gamma^0(N)$ such that $\gamma_{c_i} \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix}\right) = \gamma_{c_j} \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix}\right) \gamma$. Denote by $(\gamma_{c_i})_p$ the image of γ_{c_i} in $\mathrm{GL}_2(\mathbf{Z}_p)$, when embedded diagonally. Since c_i, c_j map to c, $(\gamma_{c_i})_p T(ord_p(N_1')) = (\gamma_{c_j})_p T(ord_p(N_1'))$ for all primes $p|N_1'$ or $p \nmid \frac{N_2'}{N_1'}$. Hence, for each of those primes there exists $g_p \in T(ord_p(N_1'))$ such that $(\gamma_{c_i})_p = (\gamma_{c_j})_p g_p$. Note that we can choose γ_{c_i} and γ_{c_j} in such a way that $\gamma_{c_i} T(\mathbf{Q}) T(\mathbf{R}) \prod_p T(\mathbf{Z}_p) = \gamma_{c_j} T(\mathbf{Q}) T(\mathbf{R}) \prod_p T(\mathbf{Z}_p)$. Hence, for primes $p|\frac{N_2'}{N_1'}$, $g_p := (\gamma_{c_j}^{-1})_p (\gamma_{c_i})_p$ is still in $T(\mathbf{Z}_p)$. This shows that $g := \gamma_{c_j}^{-1} \gamma_{c_i} \in \Gamma^0(MN)$. Now it's easy to check that $\gamma := \left(\begin{smallmatrix} 1/M & \\ & 1 \end{smallmatrix}\right) g \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix}\right)$ gives a desired $\Gamma^0(N)$-equivalence. $\square$

Put

$$H_1(dM^2, L; \Gamma^0(N)) := \mathrm{im}\left(\tilde{\phi}_{L,M}\colon \mathrm{Cl}_d(MN) \to H(dM^2, L; \Gamma^0(N))\right).$$

Remark. The map $\tilde{\phi}_{L,M}$ from $\mathrm{Cl}_d(MN)$ to $H_1(dM^2, L; \Gamma^0(N))$ is a bijection. $H_1(dM^2, L; \Gamma^0(N))$ acquires a natural group structure that makes it isomorphic to $\mathrm{Cl}_d(MN)$. Hence if Λ is any character of $\mathrm{Cl}_d(MN)$, then we can naturally think of Λ as a character of $H_1(dM^2, L; \Gamma^0(N))$.

In Chapter 4 we will naturally encounter sums like

$$\sum_{c \in \mathrm{Cl}_d(N')} \Lambda^{-1}(c) a(F, \phi_{L,M}(c))$$

for a character Λ of $\mathrm{Cl}_d(N')$. Observe that, if we denote by ρ a natural projection from $\mathrm{Cl}_d(N')$ to $\mathrm{Cl}_d(MN)$, we have the following useful fact:

$$\sum_{c\in \mathrm{Cl}_d(N')} \Lambda^{-1}(c) a(F,\phi_{L,M}(c)) = \sum_{c\in \mathrm{Cl}_d(MN)} a(F,\phi_{L,M}(c)) \sum_{\substack{\tilde{c}\in \mathrm{Cl}_d(N') \\ \rho(\tilde{c})=c}} \Lambda^{-1}(\tilde{c})$$

$$= \begin{cases} 0 & \text{if } C(\Lambda)\nmid MN \\ \frac{|\mathrm{Cl}_d(N')|}{|\mathrm{Cl}_d(MN|)} \sum_{T\in H_1(dM^2,L;\Gamma^0(N))} \Lambda^{-1}(T) a(F,T) & \text{if } C(\Lambda)|MN, \end{cases} \tag{3.5}$$

where $C(\Lambda)=\prod_p p^{c(\Lambda_p)}$ is the smallest integer such that $\Lambda|_{T_{C(\Lambda)}}=1$.

Let us now try to describe more accurately the set $H_1(dM^2,L;\Gamma^0(N))$. This will give us information on the coefficients occurring in the sum above.

Lemma 3.2.2. *Suppose that* $S'=\begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix}$ *is a matrix of discriminant* dM^2L^2 *and content* L, $\xi_{S'}=\begin{pmatrix} b'/2 & c' \\ -a' & -b'/2 \end{pmatrix}$. *Let* $E(S')$ *be the subgroup of* $\mathrm{SL}_2(\mathbf{Z})$ *defined as follows*

$$E(S') := \left\{g\in \mathrm{SL}_2(\mathbf{Z}) : g^tS'g=S'\right\}.$$

Then

1. *If* $d\neq -4,-3$, *or if* $M>1$, *then* $E(S')=\{\pm 1_2\}$.
2. *If* $(d,M)=(-4,1)$, *then* $E(S')=\{\pm 1_2, \pm\xi_{S'}\}$.
3. *If* $(d,M)=(-3,1)$, *then* $E(S')=\{\pm 1_2, \pm\left(\frac{1}{2}1_2+\xi_{S'}\right), \pm\left(-\frac{1}{2}1_2+\xi_{S'}\right)\}$.

Proof. Note that $E(S')=\{g\in T_{S'}(\mathbf{Q})\cap \mathrm{SL}_2(\mathbf{Z})\colon \det g=1\}$ and it does not depend on the content of S'. We may assume then that $L=1$. A discussion at the beginning of section 2.1 applies also when $\operatorname{disc} S'=dM^2$ and $F=\mathbf{Q}$, i.e. there is an identification

$$T_{S'}(\mathbf{Q})=\mathbf{Q}(\xi_{S'})^\times \ni x+y\xi_{S'} \longmapsto x+y\frac{\sqrt{\det \xi_{S'}}}{2} = x+y\frac{M\sqrt{d}}{2}\in \mathbf{Q}(\sqrt{d}).$$

Therefore $E(S')$ corresponds to the units of the ring of integers of $\mathbf{Q}(\sqrt{d})$ of the form $x+y\frac{M\sqrt{d}}{2}$. It's easy to check that they are of the form proposed above. □

Proposition 3.2.3. *Suppose that* $S_1,\dots,S_t$ *are matrices that are a complete set of (distinct) representatives for* $H(dM^2,L;\Gamma^0(1))$, *and* A_1,..., A_r *form a complete set of (distinct) representatives for* $\mathrm{SL}_2(\mathbf{Z})/\Gamma^0(N)$.

1. *Assume that either* $d\neq -4,-3$, *or* $M>1$. *Then* $A_i^tS_jA_i$ *gives a complete set of distinct representatives for* $H(dM^2,L;\Gamma^0(N))$, *i.e.*

$$|H(dM^2,L;\Gamma^0(N))| = tr = \frac{|\mathrm{Cl}_d(1)|}{u(d)} MN\prod_{p|M}(1-p^{-1}\left(\frac{d}{p}\right))\prod_{p|N}(1+1/p),$$

where $u(d) = 1$ *if* $d \neq -4, -3$*, and* $u(-3) = 3, u(-4) = 2$*.*

2. *Assume that* $(d, M) = (-4, 1)$ *and let* $N = 2^{n_0} p_1^{n_1} \dots p_s^{n_s}$ *be the prime decomposition of* N*. Then*

$$|H(-4, L; \Gamma^0(N))| = \frac{1}{2}\left(N \prod_{p|N}(1 + 1/p) + \mathcal{L}_{-4}\right),$$

where

$$\mathcal{L}_{-4} = \begin{cases} 2^s & \text{if } n_0 \leq 1 \text{ and } \forall_i p_i \equiv 1 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

3. *Assume that* $(d, M) = (-3, 1)$ *and let* $N = 3^{n_0} p_1^{n_1} \dots p_s^{n_s}$ *be the prime decomposition of* N*. Then*

$$|H(-3, L; \Gamma^0(N))| = \frac{1}{3}\left(N \prod_{p|N}(1 + 1/p) + 2\mathcal{L}_{-3}\right),$$

where

$$\mathcal{L}_{-3} = \begin{cases} 2^s & \text{if } n_0 \leq 1 \text{ and } \forall_i p_i \equiv 1 \pmod 6 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall (e.g. [19], Proposition 2.5 and [7], Theorem 8.2) that

$$r = N \prod_{p|N}(1 + 1/p)$$

and

$$t = |\mathrm{Cl}_d(M)| = \begin{cases} \frac{|\mathrm{Cl}_d(1)|}{u(d)} M \prod_{p|M}(1 - p^{-1}\left(\frac{d}{p}\right)) & \text{if } M > 1 \\ |\mathrm{Cl}_d(1)| & \text{if } M = 1 \end{cases}.$$

Each equivalence class in $H(dM^2, L; \Gamma^0(1))$ (i.e. $j \in \{1, \dots, t\}$ is fixed) can be written as a union of sets $\{\, {}^t g\, {}^t A_i S_j A_i g : g \in \Gamma^0(N)\}$ with $i \in \{1, \dots, r\}$. The question is whether they are all disjoint. Assume this is not the case for the sets corresponding to i_1 and i_2, i.e. assume there exists $g \in \Gamma^0(N)$ such that ${}^t A_{i_1} S_j A_{i_1} = {}^t g\, {}^t A_{i_2} S_j A_{i_2} g$. Then $S_j = {}^t(A_{i_2} g A_{i_1}^{-1}) S_j A_{i_2} g A_{i_1}^{-1}$, where $A_{i_2} g A_{i_1}^{-1} \in \mathrm{SL}_2(\mathbf{Z})$. Hence, $A_{i_2} g A_{i_1}^{-1} \in E(S_j)$ and Lemma 3.2.2 tells us precisely what these elements may be. The question is whether it does not imply $i_1 = i_2$ and how often this is the case.

1. If $d \neq -4, -3$, or if $M > 1$, then $A_{i_2} g = \pm A_{i_1}$, and so $i_1 = i_2$.

2. If $(d,M)=(-4,1)$, then either $i_1=i_2$ as above, or $A_{i_2}g=\pm\xi_{S_j}A_{i_1}$.

3. If $(d,M)=(-3,1)$, then we get two additional possibilities: either $A_{i_2}g=\pm(\frac{1}{2}I+\xi_{S_j})A_{i_1}$ or $A_{i_2}g=\pm(-\frac{1}{2}I+\xi_{S_j})A_{i_1}$.

Let us check whether the remaining cases may happen when $i_1\neq i_2$. Without loss of generality we may assume that $L=1$. Observe that both $H(-4,1;\Gamma^0(1))$ and $H(-3,1;\Gamma^0(1))$ contain only one class, namely the one determined by 1_2 and $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1\end{pmatrix}$ respectively. Indeed, each of the elements $\begin{pmatrix} a' & b'/2 \\ b'/2 & c'\end{pmatrix}$ in $H(dM^2,1;\Gamma^0(1))$ can be written uniquely in a *reduced form*, that is with $|b'|\leq a'\leq c'$ (e.g. [8], Theorem 2.8). It is easy to see that if $M=1$ and $d=-4,-3$, the only matrix satisfying these conditions is 1_2 and $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1\end{pmatrix}$ respectively. From this observation it also follows that $|\mathrm{Cl}_{-4}(1)|=|\mathrm{Cl}_{-3}(1)|=1$.

Choose a set of representatives for $\mathrm{SL}_2(\mathbf{Z})/\Gamma^0(N)$ to be[2]

$$\{A_1,\dots,A_r\}=\{\left(\begin{smallmatrix} * & u \\ * & v\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbf{Z}):v|N,u\ (\mathrm{mod}\ N/v)\}\,.$$

Let $A_{i_1}=\left(\begin{smallmatrix} * & u \\ * & v\end{smallmatrix}\right)$ and $A_{i_2}=\left(\begin{smallmatrix} * & u' \\ * & v'\end{smallmatrix}\right)$.

Assume first that $(d,M)=(-4,1)$. Then $g=\pm A_{i_2}^{-1}\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)A_{i_1}$ is in $\Gamma^0(N)$ if and only if $N|uu'+vv'$. Since $\gcd(u,v)=1=\gcd(u',v')$ and $v,v'|N$, we must have $v=\gcd(u',N)$, $v'=\gcd(u,N)$ and $\gcd(v,v')=1$. Put $u=v'\underline{u}$ and $u'=v\underline{u'}$, so that $\frac{N}{vv'}|\underline{u}\underline{u'}+1$. Note that $\underline{u'}$ is determined by $v,v',\underline{u}$. Moreover, under the assumption $\gcd(v,v')=1$, $i_1=i_2$ if and only if $v=v'=1$ and $u=u'$ satisfies $u^2=-1 \bmod N$.

Hence, if we fix u,v, then u',v' are uniquely determined and thus there are

$$\frac{N}{2}\prod_{p|N}(1+1/p)+\frac{\mathcal{L}_{-4}}{2}$$

$\Gamma^0(N)$-non-equivalent classes within each class in $H(-4,L;\Gamma^0(1))$, where

$$\mathcal{L}_{-4}:=\#\{u\in\{1,\dots,N\}:u^2=-1 \bmod N\}\,.$$

If $(d,M)=(-3,1)$, then $g=\pm A_{i_2}^{-1}(\pm\frac{1}{2}1_2+\begin{pmatrix} 1/2 & 1 \\ -1 & -1/2\end{pmatrix})A_{i_1}$ is in $\Gamma^0(N)$ if and only if

$$N|uu'+vv'+u'v \qquad \text{or} \qquad N|uu'+vv'+uv'\,. \tag{$\star\star$}$$

We will look for the solutions to the first condition, the latter one being

[2]For the proof that these indeed constitute representatives for $\mathrm{SL}_2(\mathbf{Z})/\Gamma^0(N)$, consult [19], Proposition 2.5.

symmetric.

From similar reasons as in the previous situation, $u' = v\underline{u'}$ and $\gcd(\underline{u'}, N) = 1$. Hence, $\gcd(v, v')|uu'$, and thus v and v' are coprime. Our condition becomes $\frac{N}{v}|v' + \underline{u'}(u+v)$ and implies $v' = \gcd(N, u+v)$. Let $t = u+v$ and write $t = v'\underline{t}$. It is easy to see that t runs through the rests modulo N/v and $\gcd(t, v) = 1$. Moreover, $\frac{N}{vv'}|1 + \underline{u'}\underline{t}$.

Hence, if we fix v and u, v' and u' are uniquely determined. Similarly, if $N|u\tilde{u}+v\tilde{v}+u\tilde{v}$, then $\tilde{u}$ and $\tilde{v}$ are uniquely determined by u, v. Moreover, it is easy to check that the conditions ($\star\star$) hold at the same time only if $v = v' = 1$ and $u = u'$ satisfies $u^2 + u + 1 \equiv 0 \pmod N$. Hence the conditions ($\star\star$) and uniqueness of the solution for each of them imply that unless $v = 1$ and $u^2 + u + 1 \equiv 0 \pmod N$, the matrix $\left(\begin{smallmatrix} * & u \\ * & v \end{smallmatrix}\right)$ is $\Gamma^0(N)$-equivalent to exactly two matrices. Therefore, there are

$$\frac{N}{3}\prod_{p|N}(1+1/p) + 2\frac{\mathcal{L}_{-3}}{3}$$

$\Gamma^0(N)$-non-equivalent classes within each class in $H(-3, L; \Gamma^0(1))$, where

$$\mathcal{L}_{-3} = \#\{u \in \{1, \dots, N\} : u^2 + u + 1 = 0 \bmod N\}.$$

In the following lemma we compute the quantities $\mathcal{L}_{-4}$ and $\mathcal{L}_{-3}$, and that finishes the proof. $\square$

Lemma 3.2.4.

1. *Let $N = 2^{n_0}p_1^{n_1}\dots p_s^{n_s}$ be the prime decomposition of N. Then*

$$\#\{u \in (\mathbf{Z}/N\mathbf{Z})^\times : u^2 = -1 \bmod N\} \\ = \begin{cases} 2^s & \textit{if } n_0 \le 1 \textit{ and } \forall_i p_i \equiv 1 \pmod 4 \\ 0 & \textit{otherwise.} \end{cases}$$

2. *Let $N = 3^{n_0}p_1^{n_1}\dots p_s^{n_s}$ be the prime decomposition of N. Then*

$$\#\{u \in (\mathbf{Z}/N\mathbf{Z})^\times : u^2 + u + 1 = 0 \bmod N\} \\ = \begin{cases} 2^s & \textit{if } n_0 \le 1 \textit{ and } \forall_i p_i \equiv 1 \pmod 6 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This follows from Chinese Remainder Theorem and two basic facts:

- For an odd prime p, $(\mathbf{Z}/p^n\mathbf{Z})^\times$ is a cyclic group of order $p^{n-1}(p-1)$.

- $(\mathbf{Z}/2^n\mathbf{Z})^\times$ is cyclic of order 1 and 2 for $n = 1$ and 2, respectively. If $n \geq 3$, then it is the product of two cyclic groups, one of order 2, the other of order 2^{n-2}.

1. Depending whether $p \equiv 1 \pmod 4$ or $p \equiv -1 \pmod 4$, -1 is a square in $(\mathbf{Z}/p^n\mathbf{Z})^\times$ or not, respectively. Moreover, because there is only one element of order 2, there are either 2 or 0 solutions to $u^2 = -1 \bmod p^n$. If $p = 2$, -1 is a square in $(\mathbf{Z}/2^n\mathbf{Z})^\times$ only if $n = 1$, in which case $-1 = 1^2$.

2. First note that the equation $u^2+u+1 = 0$ has no solution modulo 2. Furthermore, because the solutions are of the form $(-1+\sqrt{-3})2^{-1}$, the equation has one solution modulo 3 and no solutions modulo 3^n for $n > 1$. Now, since $u^3 - 1 = (u-1)(u^2+u+1 = 0)$, we will look for the elements of order 3 in $(\mathbf{Z}/p^n\mathbf{Z})^\times$, where $p \equiv \pm 1 \pmod 6$.

   If $p \equiv -1 \pmod 6$, then the order of $(\mathbf{Z}/p^n\mathbf{Z})^\times$ is not divisible by 3. In the other case, there are two elements of order 3, u_0 and u_0^2, say. Both of them are zeros of the polynomial $u^2+u+1 = 0 \bmod p^n$.

□

In Proposition 3.2.3 we computed the size of $H(dM^2, L; \Gamma^0(N))$. On the other hand, we know that ([35], proof of Proposition 5.3)

$$|H_1(dM^2, L; \Gamma^0(N))| = |\mathrm{Cl}_d(MN)| = \frac{|\mathrm{Cl}_d(1)|}{u(d)} MN \prod_{p|MN} (1 - p^{-1}\left(\frac{d}{p}\right))$$

if $MN > 1$, where $u(d)$ is as in Proposition 3.2.3. (If $MN = 1$, then $|H_1(dM^2, L; \Gamma^0(N))| = |\mathrm{Cl}_d(1)|$.) Combining these we get the following result.

Proposition 3.2.5. *The map* $\tilde{\phi}_{L,M} \colon \mathrm{Cl}_d(MN) \to H(dM^2, L; \Gamma^0(N))$ *is surjective if and only if* $\left(\frac{dM^2}{p}\right) = -1$ *for all primes* $p|N$.

Corollary 3.2.1. *The following conditions are equivalent:*

1. $H_1(dM^2, L; \Gamma^0(N)) = H(dM^2, L; \Gamma^0(N))$,
2. $\left(\frac{dM^2}{p}\right) = -1$ *for all* $p|N$,
3. $|H(dM^2N^2, L; \Gamma^0(1))| = |H(dM^2, L; \Gamma^0(N))|$.

Proof. This follows from the fact that $\mathrm{Cl}_d(MN) \cong H(dM^2N^2, L; \Gamma^0(1))$ (Proposition 5.3, [35]) and Proposition 3.1.1, 3.2.5. □

3.3 Discussion on the image

As we mentioned earlier, in Chapter 4 we will naturally encounter averages of Fourier coefficients of the form

$$\sum_{T \in H_1(dM^2, L; \Gamma^0(N))} \Lambda^{-1}(T) a(F, T). \tag{3.6}$$

Therefore it is important to precisely determine the set $H_1(dM^2, L; \Gamma^0(N))$. Corollary 3.2.1 describes when the sum 3.6 includes all inequivalent coefficients (according to the relation (5.2)) of given content and discriminant. However, it would be useful to know which coefficients are omitted.

Lemma 3.3.1. *Every class in $H_1(dM^2, 1; \Gamma^0(N))$ has a representative with the $(2,2)$-entry coprime to MN.*

Proof. Let $c \in \mathrm{Cl}_d(MN)$. Every class in $H_1(dM^2, 1; \Gamma^0(N))$ contains an element $\phi_{1,M}(c) = \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix}\right) S_c \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix}\right)$, where $S_c = \det(\gamma_c)^{-1}\, {}^t\gamma_c S(d) \gamma_c$ is as defined in (3.3); in particular, $t_c = \gamma_c m_c \kappa_c \in \mathrm{GL}_2(\mathbf{Q})\mathrm{GL}_2(\mathbf{R}^+)K^*_{MN}$ is a representative for c in $T(\mathbf{A})$. We will show that the $(2,2)$-entry of S_c is coprime to MN.

At each place $p|MN$, $t_{c,p} = \gamma_{c,p}\kappa_{c,p}$, so

$$S_{c,p} = \det(t_{c,p}\kappa_{c,p}^{-1})^{-1}\, {}^t(t_{c,p}\kappa_{c,p}^{-1}) S(d) t_{c,p}\kappa_{c,p}^{-1} = \det(\kappa_{c,p})\, {}^t\kappa_{c,p}^{-1} S(d) \kappa_{c,p}^{-1}\,.$$

Now, because the $(2,2)$-entry of $S(d)$ is equal to 1, and the elements on the diagonal of $\kappa_{c,p}$ are coprime to MN, the $(2,2)$-entry of $S_{c,p}$ at each place $p|MN$ is also coprime to MN. Hence the statement of the lemma. □

Even though the above lemma does not tell us precisely what the set $H_1(dM^2, 1; \Gamma^0(N))$ is, it brings an unwanted conclusion that the sum 3.6 misses *all* primitive coefficients $a(F, T)$ of paramodular forms. Indeed, the property of whether the $(2,2)$-entry of a matrix in $\mathcal{P}_2$ is coprime to N is preserved in its $\Gamma^0(N)$-equivalence class, and if T occurs in a Fourier expansion of a paramodular form of level N, its $(2,2)$-entry is always divisible by N (cf. Section 5.1.1).

Chapter 4

Main results

In this chapter we combine local and global theory to obtain our results. We incorporate the notation from Chapter 2 and 3, putting a subscript p to indicate the place at which we localise a given object.

The main result is Theorem 4.2.1, which provides a relation between Fourier coefficients of arbitrary irreducible cuspidal representation of $G(\mathbf{A})$ with trivial central character. In the next sections we specialise this result to various cuspidal Siegel modular forms of degree 2. In particular, we derive an information on non-vanishing of 'simple' Fourier coefficients (Section 4.4) and prove Maass relations for generalised Saito-Kurokawa lifts (Section 4.5).

4.1 Global Bessel models

Let d be a fundamental discriminant and $S = S(d)$ the matrix defined in (3.2). Let T, U, R be as in Chapters 2, 3. Let ψ be a fixed non trivial character of $\mathbf{A}/\mathbf{Q}$. We define the character $\theta = \theta_S$ on $U(\mathbf{A})$ by $\theta(u(X)) = \psi(\mathrm{tr}\,(SX))$. Let Λ be a character of $T(\mathbf{A})/T(\mathbf{Q})$ such that $\Lambda|_{\mathbf{A}^\times} = 1$, and denote by $\Lambda \otimes \theta$ the character of $R(\mathbf{A})$.

Throughout this chapter, let π be an (irreducible) automorphic cuspidal representation of $G(\mathbf{A})$ with trivial central character and V_π be its space of automorphic forms. For $\Phi \in V_\pi$, we define a function B_Φ on $G(\mathbf{A})$ by

$$B_\Phi(g) = \int_{\mathbf{A}^\times R(\mathbf{Q})\backslash R(\mathbf{A})} (\Lambda \otimes \theta)(r)^{-1} \Phi(rg)\, dr.$$

The $\mathbf{C}$-vector space of functions on $G(\mathbf{A})$ spanned by $\{B_\Phi : \Phi \in V_\pi\}$ is called the *global Bessel space* of type (Λ, θ_S) for π, and its vectors are called *Bessel periods*; it is invariant under the regular action of $G(\mathbf{A})$, and when the space is non-zero, the corresponding representation is a model of π, which we call a *global Bessel model* of type (Λ, θ). In fact, if there exists $\Phi \in V_\pi$ such that $B_\Phi \neq 0$, then $B_\Phi \neq 0$ for *all* $\Phi \in V_\pi$.

For $\Phi \in V_\pi$ and a symmetric matrix $S \in M_2^{\rm sym}(\mathbf{Q})$, we define the Fourier coefficient

$$\Phi_{S,\psi}(g) = \int\limits_{M_2^{\rm sym}(\mathbf{Q})\backslash M_2^{\rm sym}(\mathbf{A})} \psi^{-1}(\mathrm{tr}\,(SX))\Phi(\begin{pmatrix} 1_2 & X \\ & 1_2 \end{pmatrix} g)\,dX, \qquad g \in G(\mathbf{A}). \tag{4.1}$$

For brevity we will often shorten $\Phi_{S,\psi}$ to Φ_S.

Lemma 4.1.1.

$$B_\Phi(g) = \int_{\mathbf{A}^\times T(\mathbf{Q})\backslash T(\mathbf{A})} \Lambda^{-1}(t)\Phi_S(tg)dt \quad \textit{for } g \in G(\mathbf{A})\,.$$

Proof.

$$\begin{aligned} B_\Phi(g) &= \int_{\mathbf{A}^\times T(\mathbf{Q})\backslash T(\mathbf{A})} \int_{M_2^{\rm sym}(\mathbf{Q})\backslash M_2^{\rm sym}(\mathbf{A})} \theta_S^{-1}(u(X))\Lambda^{-1}(t)\Phi(tu(X)g)dX\,dt \\ &= \int_{\mathbf{A}^\times T(\mathbf{Q})\backslash T(\mathbf{A})} \Lambda^{-1}(t) \int_{M_2^{\rm sym}(\mathbf{Q})\backslash M_2^{\rm sym}(\mathbf{A})} \theta_{S'}^{-1}(u(Y))\Phi(u(Y)tg)dY\,dt\,, \end{aligned}$$

where $Y = (\det t)^{-1} \cdot tX\,{}^tt$ and $S' = \det t \cdot {}^tt^{-1}St^{-1} = S$. This finishes the proof. □

The next Lemma points out the importance of Fourier coefficients defined in (4.1).

Lemma 4.1.2. *Let π be a cuspidal, automorphic representation of $G(\mathbf{A})$ with trivial central character, and let $S \in M_2^{\rm sym}(\mathbf{Q})$. The following are equivalent.*

1. *$\Phi_S \neq 0$ for some $\Phi \in V_\pi$.*
2. *$\Phi_S \neq 0$ for all $\Phi \in V_\pi$.*
3. *π has a global Bessel model of type (Λ, θ_S) for some character Λ of $T(\mathbf{A})/T(\mathbf{Q})$.*

Proof. Only i) $\Rightarrow$ iii) deserves a proof. Assume that Φ_S is non-zero. Let $g \in G(\mathbf{A})$ be fixed such that $\Phi_S(g) \neq 0$. It is easy to verify that Φ_S is left $T(\mathbf{Q})$-invariant. Hence, we have a well-defined, non-zero function

$$\mathbf{A}^\times T(\mathbf{Q})\backslash T(\mathbf{A}) \longrightarrow \mathbf{C}, \qquad t \longmapsto \Phi_S(tg)\,.$$

By Fourier theory, there exists a character Λ of $T(\mathbf{Q})\backslash T(\mathbf{A})$ such that

$$\int\limits_{\mathbf{A}^\times T(\mathbf{Q})\backslash T(\mathbf{A})} \Lambda^{-1}(t)\Phi_S(tg)\,dt \neq 0\,.$$

This implies that π has a Bessel model of type (Λ, θ_S). $\square$

4.2 The key relation

Fix $\Phi \in V_\pi$, $N \in \mathbf{N}$ and assume that Φ is right invariant by the subgroup I_N of $G(\mathbf{A}_f)$ defined in Section 1.5. Let $\mathcal{S}$ be a subset of the set of primes dividing N. For any integer X we will use the subscript $X_{\mathcal{S}}$ to denote the part of X inside $\mathcal{S}$ (cf. Section 1.5).

For any two integers L, M, we define the element $H(L,M) \in G(\mathbf{A})$ by

$$H(L,M)_p := \begin{cases} \begin{pmatrix} LM^2 & & & \\ & LM & & \\ & & 1 & \\ & & & M \end{pmatrix}, & p \mid LM \\ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, & p \nmid LM \text{ or } p = \infty \end{cases}.$$

Note that $H(1,1) = 1$ and $H(L,M)_p = h_p(l,m)$ for all $p < \infty$, where $l = \mathrm{ord}_p L$ and $m = \mathrm{ord}_p M$.

Let $\Lambda = \prod_{p\le\infty} \Lambda_p$ be a character of $T(\mathbf{A})/T(\mathbf{Q})T(\mathbf{R})$ such that $\Lambda|_{\mathbf{A}^\times} = 1$ and let $C(\Lambda) = \prod_p p^{c(\Lambda_p)}$ be the smallest integer such that $\Lambda|_{T_{C(\Lambda)}} = 1$ (cf. Def. 2.2.1, 3.1.1). Suppose that $\Phi = \Phi_{\mathcal{S}} \otimes_{p\notin\mathcal{S}} \phi_p$ is a pure tensor in the space of π away from $\mathcal{S}$. For each prime $p \notin \mathcal{S}$ assume that

- π_p has no local Bessel model of type (Λ_p, θ_p)

or (4.2)

- π_p has a local Bessel model and $m_{\phi_p,\Lambda_p} < \infty$.

Remark. The point of introducing the set $\mathcal{S}$ is so that we may formulate our results without bothering about the value, even finiteness, of m_{ϕ_p,Λ_p}. If (4.2) holds for each prime $p|N$ and we know a nice formula for m_{ϕ_p,Λ_p} for each p, we may assume that $\mathcal{S} = \emptyset$. This is the case for almost every representation listed in Table 2 - check further the remark at the end of Chapter 2.

If π has a global Bessel model of type (Λ, θ), then for each place p of $\mathbf{Q}$, π_p has a local Bessel model of type (Λ_p, θ_p) and each ϕ_p corresponds to a (unique up to multiples) vector B_{ϕ_p} in the local Bessel model of π_p.

Definition 4.2.1. For each $\Phi \in V_\pi$ and a character Λ define

$$M^{\mathcal{S}}_{\Phi,\Lambda} := \begin{cases} 1 & \text{if } \pi_p \text{ has no local Bessel model for some } p \notin \mathcal{S} \\ \prod_{p\notin\mathcal{S}} p^{m_{\phi_p,\Lambda_p}} & \text{otherwise.} \end{cases} \tag{4.3}$$

Remark. If π_p is of type I, then $m_{\phi_p,\Lambda_p} = c(\Lambda_p)$ by Sugano's theorem 2.2.1. The other spherical representations (type IIb, IIIb, IVd, Vd, VId) with trivial central character admit local Bessel model if and only if $\Lambda = 1$. In this case $m_{\phi_p,\Lambda_p} = 0$.

The following lemma is the base for our main results.

Lemma 4.2.1. *Let L, M, L', M' be positive integers such that*

- $L'|L$*,* $L'_{\mathcal{S}} = L_{\mathcal{S}}$
- $M^{\mathcal{S}}_{\Phi,\Lambda}|M'|M$*,* $M'_{\mathcal{S}} = M_{\mathcal{S}}$*.*

If a local Bessel model exists at all $p \notin \mathcal{S}$, then the following relation holds:

$$B_\Phi(H(L,M)) \prod_{\substack{p|L'M' \\ p\notin\mathcal{S}}} B_{\phi_p}(h_p(l',m')) = B_\Phi(H(L',M')) \prod_{\substack{p|LM \\ p\notin\mathcal{S}}} B_{\phi_p}(h_p(l,m))\,. \tag{4.4}$$

Otherwise, $B_\Phi(H(\underline{L},\underline{M})) = 0$ for all integers $\underline{L},\underline{M}$.

Proof. Observe first that if π does not have a global Bessel model of type (Λ,θ), then both sides of (4.4) are zero. We may assume then that the global Bessel model exists.

By uniqueness of local Bessel functionals,

$$B_\Phi(H(L,M)) = C\prod_{p\in\mathcal{S}} B_{\phi_p}(h_p(l,m)) \prod_{p\notin\mathcal{S}} B_{\phi_p}(h_p(l,m))$$

for any positive integers L, M. The constant C can be found if we specialise to $L_{\mathcal{S}}$ and $M_{\mathcal{S}}M^{\mathcal{S}}_{\Phi,\Lambda}$:

$$B_\Phi(H(L_{\mathcal{S}}, M_{\mathcal{S}}M^{\mathcal{S}}_{\Phi,\Lambda})) = C\prod_{p\in\mathcal{S}} B_{\phi_p}(h_p(l,m)) \prod_{p\notin\mathcal{S}} 1$$

(recall that we set the normalisation $B_{\phi_p}(h_p(0, m_{\phi_p,\Lambda_p})) = 1$ for $p \notin \mathcal{S}$), and thus

$$B_\Phi(H(L,M)) = B_\Phi(H(L_{\mathcal{S}}, M_{\mathcal{S}}M^{\mathcal{S}}_{\Phi,\Lambda})) \prod_{\substack{p|LM \\ p\notin\mathcal{S}}} B_{\phi_p}(h_p(l,m))\,.$$

Using this equation with L', M' as in the statement of the lemma, we obtain the relation (4.4). Note that without the assumption $M^{\mathcal{S}}_{\Phi,\Lambda}|M'$ the statement is still true, but we have zeros on both sides of the equality. $\square$

From this simple looking relation (4.4) we obtain a correspondence between the Fourier coefficients (4.1) that will be crucial for our applications. We start with an auxiliary lemma.

Lemma 4.2.2. *Let $A \in \mathrm{GL}_2(\mathbf{Q}), \alpha \in \mathbf{Q}^\times, \gamma = \left(\begin{smallmatrix} A & \\ & \alpha\, {}^tA^{-1} \end{smallmatrix}\right)$, and let γ_f be the image of γ in $G(\mathbf{A}_f)$. Then for any automorphic form Φ on $G(\mathbf{A})$, any matrix $T \in M_2^{\mathrm{sym}}(\mathbf{Q})$ and $g_\infty \in G(\mathbf{R})^+$ we have*

$$\Phi_T(g_\infty \gamma_f) = \Phi_{\alpha^{-1}\, {}^tATA}(\gamma_\infty^{-1} g_\infty)\,,$$

where $\gamma_\infty = \gamma\gamma_f^{-1}$.

Proof. Using the fact that Φ is left $G(\mathbf{Q})$-invariant and the substitution $X = \alpha^{-1} AY\, {}^tA$, we obtain

$$\begin{aligned}
\Phi_T(g_\infty \gamma_f) &= \int_{M_2^{\mathrm{sym}}(\mathbf{Q})\backslash M_2^{\mathrm{sym}}(\mathbf{A})} \psi^{-1}(\mathrm{tr}\,(TX))\Phi(u(X)g_\infty\gamma_f)dX \\
&= \int_{M_2^{\mathrm{sym}}(\mathbf{Q})\backslash M_2^{\mathrm{sym}}(\mathbf{A})} \psi^{-1}(\mathrm{tr}\,(TX))\Phi(u(X)\gamma\gamma_\infty^{-1}g_\infty)dX \\
&= \int_{M_2^{\mathrm{sym}}(\mathbf{Q})\backslash M_2^{\mathrm{sym}}(\mathbf{A})} \psi^{-1}(\mathrm{tr}\,(TX))\Phi(u(\alpha A^{-1}X\, {}^tA^{-1})\gamma_\infty^{-1}g_\infty)dX \\
&= \Phi_{\alpha^{-1}\, {}^tATA}(\gamma_\infty^{-1} g_\infty)\,.
\end{aligned}$$

$\square$

Theorem 4.2.1. *Let π be an irreducible automorphic cuspidal representation of $G(\mathbf{A})$ with trivial central character and $\Phi \in V_\pi$ an automorphic form in its vector space. Assume that Φ is right I_N-invariant for some $N \in \mathbf{N}$ and let $\mathcal{S}$ be a subset of the set of primes dividing N. Let $S = S(d)$, and ψ be a fixed non trivial character of $\mathbf{A}/\mathbf{Q}$. Let Λ be a character of $T(\mathbf{A})/T(\mathbf{Q})T(\mathbf{R})$ such that $\Lambda|_{\mathbf{A}^\times} = 1$ and (4.2) holds. Then for any L, M, L', M' satisfying the conditions of Lemma 4.2.1 and such that $C(\Lambda)|M'N$, we have the following correspondence between the Fourier coefficients* (4.1)*:*

$$\frac{|\mathrm{Cl}_d(M'N)|}{|\mathrm{Cl}_d(MN)|} \sum_{c\in \mathrm{Cl}_d(MN)} \Lambda^{-1}(t_c) \int\limits_{\mathbf{R}^\times\backslash T(\mathbf{R})} \Phi_{\phi_{L,M}(c)}(H_\infty^{-1} m_c t_\infty)dt_\infty \prod_{\substack{p|L'M' \\ p\notin\mathcal{S}}} B_{\phi_p}(h_p(l',m'))$$

$$= \sum_{c\in \mathrm{Cl}_d(M'N)} \Lambda^{-1}(t_c) \int\limits_{\mathbf{R}^\times\backslash T(\mathbf{R})} \Phi_{\phi_{L',M'}(c)}((H'_\infty)^{-1}m_c t_\infty)dt_\infty \prod_{\substack{p|LM\\ p\notin\mathcal{S}}} B_{\phi_p}(h_p(l,m)) \,, \tag{4.5}$$

where $t_c \in T(\mathbf{A})$ are representatives for c in $\mathrm{Cl}_d(MN)$ or $\mathrm{Cl}_d(M'N)$ such that $t_c = \gamma_c m_c \kappa_c$ by strong approximation theorem (cf. Section 3.1), $\phi_{L,M}(c)$ is defined as in (3.4)*,*

$$H_\infty := \begin{pmatrix} LM^2 &&& \\ & LM && \\ && 1 & \\ &&& M \end{pmatrix} \quad \textit{and} \quad H'_\infty = \begin{pmatrix} L'M'^2 &&& \\ & L'M' && \\ && 1 & \\ &&& M' \end{pmatrix}.$$

Proof. In view of the relation (4.4), it suffices to compute the values $B_\Phi(H(L,M))$.

Let $\Phi^{L,M}(g) := \Phi(gH(L,M))$ for $g \in G(\mathbf{A})$; because Φ is right I_N-invariant, $\Phi^{L,M}$ is right invariant by

$$H_\infty I_N H_\infty^{-1} = \{\prod_{p<\infty} \begin{pmatrix} * & MN* & LM^2* & LM* \\ */M & * & LM* & L* \\ *N/LM^2 & *N/LM & * & */M \\ *N/LM & *N/L & MN* & * \end{pmatrix} \in G(\mathbf{Q}_p) : * \in \mathbf{Z}_p\},$$

where $H_\infty := \begin{pmatrix} LM^2 &&& \\ & LM && \\ && 1 & \\ &&& M \end{pmatrix} \in G(\mathbf{R})^+$. In particular, $\Phi^{L,M}$ is right invariant by T_{MN} and K^*_{MN} (defined at the beginning of Chapter 3).

Following the notation of Section 3.1, we can write

$$T(\mathbf{A}) = \coprod_{c\in\mathrm{Cl}_d(MN)} t_c T(\mathbf{Q})T(\mathbf{R})T_{MN}\,,$$

where we choose $t_c \in \prod_{p<\infty} T(\mathbf{Q}_p)$, and by strong approximation theorem write $t_c = \gamma_c m_c \kappa_c$ with $\gamma_c \in \mathrm{GL}_2(\mathbf{Q})$, $m_c \in \mathrm{GL}_2(\mathbf{R})^+$ and $\kappa_c \in K^*_{MN}$.

With this preparation we are ready to compute the values $B_\Phi(H(L,M))$.

$$\begin{aligned} B_\Phi(H(L,M)) &= \int_{\mathbf{A}^\times T(\mathbf{Q})\backslash T(\mathbf{A})} \Lambda^{-1}(t)\Phi_S^{L,M}(t)dt \\ &= \sum_{c\in\mathrm{Cl}_d(MN)} \int_{\mathbf{R}^\times\backslash T(\mathbf{R})} \Phi_S^{L,M}(t_c t_\infty)dt_\infty \int_{\mathbf{A}_f^\times T(\mathbf{Q})\cap T_{MN}\backslash T_{MN}} \Lambda^{-1}(t_c t_\infty t_{MN})dt_{MN} \end{aligned}$$

Observe that if $C(\Lambda) \nmid MN$, then the inner integral is equal to zero and the equation (4.5) holds. Henceforth we assume that $C(\Lambda)|MN$. With this assumption and using Lemma 4.2.2 twice, we have

$$B_\Phi(H(L,M)) = \frac{1}{|\mathrm{Cl}_d(MN)|} \sum_{c\in\mathrm{Cl}_d(MN)} \Lambda^{-1}(t_c) \int_{\mathbf{R}^\times\backslash T(\mathbf{R})} \Phi_S^{L,M}(\gamma_c m_c t_\infty)dt_\infty$$

$$= \frac{1}{|\mathrm{Cl}_d(MN)|} \sum_{c\in \mathrm{Cl}_d(MN)} \Lambda^{-1}(t_c) \int_{\mathbf{R}^\times\backslash T(\mathbf{R})} \Phi_S^{L,M}(t_\infty(\gamma_c)_f)dt_\infty$$
$$= \frac{1}{|\mathrm{Cl}_d(MN)|} \sum_{c\in \mathrm{Cl}_d(MN)} \Lambda^{-1}(t_c) \int_{\mathbf{R}^\times\backslash T(\mathbf{R})} \Phi_{S_c}^{L,M}(m_c t_\infty)dt_\infty$$
$$= \frac{1}{|\mathrm{Cl}_d(MN)|} \sum_{c\in \mathrm{Cl}_d(MN)} \Lambda^{-1}(t_c) \int_{\mathbf{R}^\times\backslash T(\mathbf{R})} \Phi_{\phi_{L,M}(c)}(H_\infty^{-1} m_c t_\infty)dt_\infty\,,$$

where $\phi_{L,M}(c)$ is defined as in (3.4). □

Corollary 4.2.1. *Suppose that Φ is right invariant by $I_{N_1,N}$ for some $N_1|N$. Then we get a simpler formula, where the sum runs over the elements of the ray class group $\mathrm{Cl}_d(MN_1)$ and it suffices to require $C(\Lambda)|M'N_1$:*

$$\frac{|\mathrm{Cl}_d(M'N_1)|}{|\mathrm{Cl}_d(MN_1)|} \sum_{c\in \mathrm{Cl}_d(MN_1)} \Lambda^{-1}(t_c)\int_{\mathbf{R}^\times\backslash T(\mathbf{R})} \Phi_{\phi_{L,M}(c)}(H_\infty^{-1} m_c t_\infty)dt_\infty \prod_{\substack{p|L'M'\\ p\notin\mathcal{S}}} B_{\phi_p}(h_p(l',m'))$$
$$= \sum_{c\in \mathrm{Cl}_d(M'N_1)} \Lambda^{-1}(t_c) \int_{\mathbf{R}^\times\backslash T(\mathbf{R})} \Phi_{\phi_{L',M'}(c)}((H'_\infty)^{-1} m_c t_\infty)dt_\infty \prod_{\substack{p|LM\\ p\notin\mathcal{S}}} B_{\phi_p}(h_p(l,m))\,.$$

The rest of notation is as in Theorem 4.2.1.

In the next section we specialise the relation (4.5) to automorphic forms that give rise to Siegel modular forms and obtain very general relations between their Fourier coefficients. This will lead to further applications.

4.3 Application to Siegel modular forms

Let $\pi = \otimes_p \pi_p$ be an irreducible automorphic cuspidal representation of $G(\mathbf{A})$ with trivial central character and such that $\pi_\infty = L(k,k)$, the lowest weight representation of scalar minimal K-type of weight k. Let Φ be an automorphic form in the space of π and let ϕ_∞ be a lowest weight vector of π_∞. This means that

$$\Phi(gk_\infty) = j(k_\infty, i_2)^{-k}\Phi(g) \qquad \text{for all} \quad k_\infty\in K_\infty,\, g\in G(\mathbf{A})\,, \qquad (4.6)$$

where

(i) K_∞ is a maximal compact subgroup of $G(\mathbf{R})^+$ such that any k_∞ in K_∞ is of the form $\left(\begin{smallmatrix} A & B\\ -B & A\end{smallmatrix}\right)$,

(ii) $i_2 := i1_2$,

(iii) $j((\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}), i_2) = \det(Ci_2 + D)$.

Furthermore, assume that there are integers N_1, N_2 with $N_1|N_2$ such that Φ is right invariant by I_{N_1,N_2}. Let $\mathcal{S}$ be a subset of the set of primes dividing N_2 such that (4.2) holds for each prime $p \notin \mathcal{S}$.

Now, define

$$F(Z) := \Phi(g)j(g, i_2)^k \mu(g)^{-k}\,, \tag{4.7}$$

where $g \in G(\mathbf{R})$ is such that $g \cdot i_2 = Z$ and

$$(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \cdot i_2 := (Ai_2 + B)(Ci_2 + D)^{-1}\,.$$

Such a function F is holomorphic and satisfies

$$\forall_{\gamma=(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix})\in\Gamma_0(N_1,N_2)}\, F|_k\gamma(Z) := \mu(\gamma)^k j(\gamma, Z)^{-k} F(\gamma \cdot Z) = F(Z)\,; \tag{4.8}$$

it is a cuspidal *Siegel modular form* of degree 2, level $\Gamma_0(N_1, N_2)$ and weight k that is an eigenform of the local Hecke algebra at all primes $p \nmid N_2$.[1] It follows that F admits a unique Fourier expansion

$$F(Z) = \sum_{T\in\mathcal{P}_2} a(F, T)e(\mathrm{tr}\,(TZ))\,, \quad \text{where} \quad e(x) := e^{2\pi i x}\,,$$

where the sum runs over the matrices in the set $\mathcal{P}_2$ defined in (3.1).

Observe that the correspondence (4.7) is bijective in a sense that to any Siegel cusp form F that satisfies the above conditions and gives rise to an irreducible representation we can attach an automorphic form Φ via

$$\Phi(g) := F|_k g_\infty(i_2)\,, \quad g = g_{\mathbf{Q}} g_\infty g_0 \in G(\mathbf{Q})G(\mathbf{R})^+ I_{N_1,N_2} = G(\mathbf{A})\,; \tag{4.9}$$

Φ is called the *adelisation* of F.

With this preparation we may begin our way to formulating a version of Corollary 4.2.1 for Siegel modular forms. For the rest of this section we assume that $\psi = \prod_{p\leq\infty} \psi_p$ is the character of $\mathbf{A}/\mathbf{Q}$ such that

- the conductor of ψ_p is $\mathbf{Z}_p$ for all $p < \infty$,
- $\psi_\infty(x) = e(x)$ for $x \in \mathbf{R}$.

Lemma 4.3.1. *Let Φ be an automorphic form on $G(\mathbf{A})$ that satisfies the equation* (4.6)*, and let F be as in* (4.7)*. Then for any matrix $T \in M_2^{\mathrm{sym}}(\mathbf{Q})$*

[1] In fact, these functions, coming from irreducible representations, span the space of Siegel cusp forms of degree 2, level $\Gamma_0(N_1, N_2)$ and weight k.

and $g_\infty \in G(\mathbf{R})^+$ *we have*

$$\Phi_T(g_\infty) = \mu(g_\infty)^k j(g_\infty, i_2)^{-k} a(F,T) e(\mathrm{tr}\,(T(g_\infty \cdot i_2)))\,.$$

Proof. It is easy to check that the automorphy factor j defined above has the property

$$j(g_1 g_2, Z) = j(g_1, g_2 \cdot Z) j(g_2, Z)\,.$$

Hence,

$$\begin{aligned}
\Phi_T(g_\infty) &= \int_{M_2^{\mathrm{sym}}(\mathbf{Q})\backslash M_2^{\mathrm{sym}}(\mathbf{A})} \psi^{-1}(\mathrm{tr}\,(TX))\Phi(u(X)g_\infty)dX \\
&= \int_{M_2^{\mathrm{sym}}(\mathbf{Z})\backslash M_2^{\mathrm{sym}}(\mathbf{R})} e(-\mathrm{tr}\,(TX))\mu(g_\infty)^k j(u(X)g_\infty, i_2)^{-k} \\
&\qquad\qquad F(u(X)g_\infty \cdot i_2)dX \\
&= \mu(g_\infty)^k j(g_\infty, i_2)^{-k} \sum_{T'} a(F,T') \\
&\qquad \int_{M_2^{\mathrm{sym}}(\mathbf{Z})\backslash M_2^{\mathrm{sym}}(\mathbf{R})} e(-\mathrm{tr}\,(TX)) e(\mathrm{tr}\,(T'(g_\infty \cdot i_2))) e(\mathrm{tr}\,(T'X))dX \\
&= \mu(g_\infty)^k j(g_\infty, i_2)^{-k} a(F,T) e(\mathrm{tr}\,(T(g_\infty \cdot i_2)))\,.
\end{aligned}$$

□

Theorem 4.3.1. *Let* π, Φ, F *and* N_1, N_2 *be as above. Let* $S = S(d)$ *and let* Λ *be a character of* $T(\mathbf{A})/T(\mathbf{Q})T(\mathbf{R})$ *such that* $\Lambda|_{\mathbf{A}^\times} = 1$. *Let* $\mathcal{S}$ *be a subset of the set* $\{p : p|N_2\}$ *such that* (4.2) *holds for all* $p \notin \mathcal{S}$. *Then for any* L, M, L', M' *satisfying the conditions of Lemma 4.2.1 and such that* $C(\Lambda)|M'N_1$, *we have the following relation between the Fourier coefficients of* F*:*

$$\begin{aligned}
\frac{|\mathrm{Cl}_d(M'N_1)|}{|\mathrm{Cl}_d(MN_1)|}\left(\frac{L'M'}{LM}\right)^k & \sum_{T \in H_1(dM^2, L; \Gamma^0(N_1))} \Lambda^{-1}(T) a(F,T) \prod_{\substack{p|L'M' \\ p \notin \mathcal{S}}} B_{\phi_p}(h_p(l', m')) \\
&= \sum_{T' \in H_1(dM'^2, L'; \Gamma^0(N_1))} \Lambda^{-1}(T') a(F,T') \prod_{\substack{p|LM \\ p \notin \mathcal{S}}} B_{\phi_p}(h_p(l, m))\,. \qquad (4.10)
\end{aligned}$$

Proof. We specialise Corollary 4.2.1 to Φ that is the adelisation of F. It is enough to compute the following integral.

$$\begin{aligned}
&\int_{\mathbf{R}^\times\backslash T(\mathbf{R})} \Phi_{\phi_{L,M}(c)}(H_\infty^{-1} m_c t_\infty) dt_\infty \\
&\qquad \overset{\text{Lemma 4.3.1}}{=} (LM)^{-k} a(F, \phi_{L,M}(c))
\end{aligned}$$

$$\begin{aligned}
&\cdot \int_{\mathbf{R}^\times \backslash T(\mathbf{R})} e(\mathrm{tr}\,(\phi_{L,M}(c)(H_\infty^{-1} m_c t_\infty \cdot i_2)))dt_\infty \\
&= (LM)^{-k} a(F, \phi_{L,M}(c)) \int_{\mathbf{R}^\times \backslash T(\mathbf{R})} e(\mathrm{tr}\,(S_c(m_c t_\infty \cdot i_2)))dt_\infty \\
&= (LM)^{-k} a(F, \phi_{L,M}(c)) \int_{\mathbf{R}^\times \backslash T(\mathbf{R})} e(\mathrm{tr}\,(S i_2))dt_\infty \\
&= r(LM)^{-k} a(F, \phi_{L,M}(c)) e^{-2\pi \mathrm{tr}\, S},
\end{aligned}$$

where $r = \int_{\mathbf{R}^\times \backslash T(\mathbf{R})} dt_\infty$. Now, recall from Chapter 3 that

$$\{\phi_{L,M}(c) \colon c \in \mathrm{Cl}_d(MN_1)\} = H_1(dM^2, L; \Gamma^0(N_1)) .$$

□

Remark. Theorem 4.3.1 imposes two extra conditions on M' that depend on Λ: $M^{\mathcal{S}}_{\Phi,\Lambda} | M'$ and $C(\Lambda) | M' N_1$. However, thanks to Lemma 2.3.1 and Sugano's theorem 2.2.1, we only need to assume that $M^{\mathcal{S}}_{\Phi,\Lambda} | M'$ and

$$c(\Lambda_p) \le m_{\phi_p, \Lambda_p} + n_{1,p} \quad \text{for} \quad p \in \mathcal{S} .$$

So, in particular, if we may take $\mathcal{S}$ to be the emptyset (as is the case when, for example, F has level $\Gamma_0(N_1, N_2)$ with N_1, N_2 square-free, and at primes $p | N_2$, π_p, Λ_p do not satisfy two very specific conditions stated in Corollary 4.4.2), we may reduce the conditions on M' to $M^{\mathcal{S}}_{\Phi,\Lambda} | M'$.

Remark. The proofs of Theorem 4.2.1 and 4.3.1 also provide the equality

$$B_\Phi(H(L,M)) = \begin{cases} \frac{r(LM)^{-k}}{|\mathrm{Cl}_d(MN_1)|} e^{-2\pi \mathrm{tr}\, S} \sum\limits_{T \in H_1(dM^2, L; \Gamma^0(N_1))} \Lambda^{-1}(T) a(F, T), & C(\Lambda) | MN_1 \\ 0, & \text{otherwise}, \end{cases}$$

where r is the non-zero constant depending only on S and the choice of normalisation for the Haar measure.

If in Theorem 4.3.1 we take as $\mathcal{S}$ the set of all primes dividing N_2, then Sugano's theorem ensures that $M^{\mathcal{S}}_{\Phi,\Lambda} = \prod_{p \nmid N_2} p^{ord_p C(\Lambda)}$, and thus we can simplify Theorem 4.3.1 to the following form:

Corollary 4.3.1. *Let F be a Siegel cusp form of degree 2, level $\Gamma_0(N_1, N_2)$ and weight k. Suppose that F is an eigenform of the local Hecke algebra at all primes $p \nmid {N_2}^2$. Let d be a fundamental discriminant and let*

[2] It is enough to assume that F is an eigenform of the Hecke operators $T(p)$ and $T(p^2)$ at $p \nmid N_2$. For the definition of these Hecke operators see for example [18], Chapter 5 or [1].

L, M, L', M' be positive integers such that $L'|L$, $M'|M$, $(L, N_2^\infty) = (L', N_2^\infty)$ and $(M, N_2^\infty) = (M', N_2^\infty)$ (cf. Section 1.5). Then for all characters Λ of $H_1(dM'^2, L, \Gamma^0(N_1)) \cong \mathrm{Cl}_d(M'N_1)$,

$$\frac{|\mathrm{Cl}_d(M'N_1)|}{|\mathrm{Cl}_d(MN_1)|}\left(\frac{L'M'}{LM}\right)^k \sum_{T\in H_1(dM^2,L;\Gamma^0(N_1))} \Lambda^{-1}(T)a(F,T) \prod_{\substack{p|L'M'\\ p\nmid N_2}} B_{\phi_p}(h_p(l',m'))$$
$$= \sum_{T'\in H_1(dM'^2,L';\Gamma^0(N_1))} \Lambda^{-1}(T')a(F,T') \prod_{\substack{p|LM\\ p\nmid N_2}} B_{\phi_p}(h_p(l,m))\,. \quad (4.11)$$

Proof. This follows immediately from Theorem 4.3.1 (for $\mathcal{S} = \{p : p|N_2\}$) and Proposition 3.11, [50], which states that in our setting the following conditions are equivalent:

(i) F is an eigenform of the local Hecke algebra at all primes $p \nmid N_2$.

(ii) If π', π'' are two irreducible cuspidal representations both of which occur as subrepresentations of the representation π associated with F, then $\pi'_p \cong \pi''_p$ for all primes $p \nmid N_2$.

As a result, $F = \sum_i F_i$, where each F_i has the same local data at $p \nmid N_2$ and is as in Theorem 4.3.1. □

4.4 Applications to non-vanishing of Fourier coefficients

Formula (4.10) yields many relations between Fourier coefficients of Siegel modular forms that are invariant under the action of $\Gamma_0(N_1, N_2)$. It gives us, among others, an insight on the existence of Fourier coefficients $a(F, T)$, where *both* content and discriminant of the matrix T are smallest possible.[3]

Proposition 4.4.1. *Let Φ, F, N_1, N_2 be as in Theorem 4.3.1, and let L, M, d be such that $a(F, T_0) \neq 0$ for some $T_0 \in H_1(dM^2, L; \Gamma^0(N_1))$. Then there exists a character Λ of $\mathrm{Cl}_d(MN_1) \cong H_1(dM^2, L; \Gamma^0(N_1))$ such that*

$$\sum_{T\in H_1(dM^2,L;\Gamma^0(N_1))} \Lambda^{-1}(T)a(F,T) \neq 0\,. \quad (4.12)$$

Suppose that $\mathcal{S}$ is a set of primes dividing N_2 such that ϕ_p is optimal for all $p \notin \mathcal{S}$ (cf. Definition 2.3.2). Then for any character Λ satisfying (4.12)*,*

[3] The formulas that one may obtain from the action of Hecke operators allow only to reduce the content of T.

we have $M_{\Phi,\Lambda}^{\mathcal{S}} M_{\mathcal{S}} | M$ and there exists a $T \in H_1(d(M_{\Phi,\Lambda}^{\mathcal{S}} M_{\mathcal{S}})^2, L_{\mathcal{S}}; \Gamma^0(N_1))$ so that $a(F,T) \neq 0$. In particular, if $L_{\mathcal{S}} = 1$, then $a(F,T) \neq 0$ for a primitive matrix T.

Proof. This is a direct corollary from Theorem 4.3.1. If $a(F,T_0) \neq 0$ for some $T_0 \in H_1(dM^2, L; \Gamma^0(N_1))$, then there exists a character Λ of $H_1(dM^2, L; \Gamma^0(N_1))$ that satisfies (4.12). By the remark from previous section, it also follows that $B_\Phi(H(L,M)) \neq 0$. Let $\mathcal{S}$ be as stated above and choose $M' = M_{\Phi,\Lambda}^{\mathcal{S}} M_{\mathcal{S}}$. Then, as explained in the previous section, all the conditions of Theorem 4.3.1 hold. Now, because $B_{\phi_p}(h_p(0, m'_p)) \neq 0$ for all $p | M'$ outside $\mathcal{S}$ (where $m'_p := ord_p M'$), the left hand side of (4.10) is non-zero for $L' = L_{\mathcal{S}}$. The hypothesis follows. $\square$

If we choose $\mathcal{S}$ to be simply the set of all primes dividing N_2, then ϕ_p is automatically optimal for all primes outside $\mathcal{S}$. Furthermore, using the argument of Corollary 4.3.1, it suffices to assume that F is an eigenform of the local Hecke algebra at primes not dividing N_2. Hence, we get the following result:

Corollary 4.4.1. *Let F be a cuspidal Siegel modular form of degree 2, level $\Gamma_0(N_1, N_2)$ and weight k. Assume that F is an eigenform of the local Hecke algebra at all primes $p \nmid N_2$. Suppose that $a(F,T_0) \neq 0$ for some $T_0 \in \bigcup_{d,M,L} H_1(dM^2, L; \Gamma^0(N_1))$ (automatic if $N_1 = 1$). Then $a(F,T) \neq 0$ for some T such that* $\operatorname{cont} T | N_2^\infty$.

Observe that in the case $N_1 = 1$, this corollary is a special case of a theorem due to Yamana [62]; the new case is $N_1 > 1$. Yamana's theorem and similar results (due to Zagier [63], Ibukiyama and Katsurada [18]) played a crucial role in theorems of Saha and Schmidt ([51], [52], [49]), which concerned determination of Siegel modular forms of degree 2 by fundamental Fourier coefficients. In the next chapter we provide more information on this topic and emphasize the importance of non-vanishing of a Fourier coefficient $a(F,T)$ with content of T smallest possible on the example of paramodular forms. The proof will be carried out using classical methods, but we hope to extend it in the future basing on the results in this chapter.

The advantage of our proof, in comparison to the one given by Yamana[4], is a usage of representation-theoretical structure. Thanks to this, Corollary 4.4.1 may be improved or extended alongside the development

[4] Yamana generalised a method of Zagier [63], which based on the Taylor expansion of the Fourier-Jacobi coefficients.

in representation theory. A good example of this interplay is the following result, which makes use of Theorems 2.3.1, 2.3.2, 2.3.3.

Corollary 4.4.2. *Let F be a cuspidal Siegel modular form of degree 2, weight k, level $\Gamma_0(N_1, N_2)$ with N_1, N_2 square-free. Suppose that the adelisation of F generates an irreducible automorphic representation π. Assume also that F is an eigenform of the $U(p)$-operator for all primes $p|\frac{N_2}{N_1}$ and an eigenform of the local Hecke algebra at $p \nmid N_2$*[5]*. Suppose that F has a non-zero Fourier coefficient $a(F, T_0)$ for some $T_0 \in H_1(dM^2, L; \Gamma^0(N_1))$ and Λ is such that* (4.12) *holds. Assume moreover that* none *of the following specific conditions holds at $p|N_2$:*

- *π_p is of type IVa, $c(\Lambda_p) = 1$, $\left(\frac{\mathbf{Q}_p(\sqrt{d})}{p}\right) = 1$ and $\Lambda_p(1, p) = \sigma(\varpi)$ (c.f. Theorem 2.3.1);*
- *π_p is of type IIa, $\left(\frac{\mathbf{Q}_p(\sqrt{d})}{p}\right) = 1$ and $\Lambda_p(1, p) = -\omega_p = \Lambda_p(p, 1)$ (c.f. Theorem 2.3.2).*

Then F has a non-vanishing primitive Fourier coefficient, i.e. $a(F, T) \neq 0$ for some matrix T with $\operatorname{cont} T = 1$. *Moreover, if $M|N_1$ and we can choose Λ so that $c(\Lambda_p) \leq 1$ for all $p|M$, then F has a non-zero fundamental Fourier coefficient.*

Proof. This follows from a combination of Proposition 4.4.1 and Theorems 2.3.1, 2.3.2, 2.3.3, which allow us to take $\mathcal{S} = \emptyset$. □

The above results rely on a seed matrix $T_0 \in H_1(dM^2, L; \Gamma^0(N_1))$ such that $a(F, T_0) \neq 0$. The question naturally arises: does such a T_0 always exist? In this context we present two results. The first one follows from Corollary 3.2.1.

Proposition 4.4.2. *Let $F \neq 0$ be a cuspidal Siegel modular form of level $\Gamma_0(1, N_2)$, weight k. Then there exist a fundamental discriminant d and integers L, M such that $a(F, T_0) \neq 0$ for some matrix $T_0 \in H_1(dM^2, L; \Gamma^0(1))$.*

The second one relies on a well-known construction called Siegelisation.

Definition 4.4.1. Fix positive integers N_1, N_2 such that $N_1|N_2$, and let F be a cuspidal Siegel modular form of level $\Gamma_0(N_1, N_2)$, weight k. Define the

[5] As in Corollary 4.3.1, at primes $p \nmid N_2$ it is enough to assume that F is an eigenform of the Hecke operators $T(p), T(p^2)$; for the definition of the $U(p)$-operator see [52]. In particular, the conditions on F in Corollary 4.4.2 imply that F is a newform in the sense of [9].

Siegelisation F' of F to be the Siegel modular form of level $\Gamma_0(1, N_2)$ given by

$$F' := \sum_{\gamma \in \Gamma^0(N_1)\backslash \mathrm{SL}_2(\mathbf{Z})} F|_k \begin{pmatrix} \gamma & \\ & {}^t\gamma^{-1} \end{pmatrix}. \tag{4.13}$$

Proposition 4.4.3. *Let F be a cuspidal Siegel modular form of weight k and level $\Gamma_0(N_1, N_2)$, and F' its Siegelisation.*

1. *Suppose F is an eigenform for the Hecke algebra at some prime p, $p \nmid N_1$. Then so is F'.*

2. *The adelisations of F and F' generate the same automorphic representation.*

3. *Assume that F is an eigenform of the local Hecke algebra at all primes $p \nmid N_2$. Suppose $a(F', T) \neq 0$ for $T \in H(dM^2, L'; \Gamma^0(1))$. Then $a(F, T_0) \neq 0$ for some $T_0 \in H(dM^2, L; \Gamma^0(N_1))$, where $L = (L', N_2^\infty)$.*

Proof. Let Φ, Φ' be the adelisations of F, F' respectively. By the definition (4.9) of the adelisation, it is clear that at each place $p \nmid N_1$, the group $G(\mathbf{Q}_p)$ acts on Φ and Φ' in the same way. Hence the property of being an eigenform for the local Hecke algebra at $p \nmid N_1$ is preserved by the Siegelisation.

As we mentioned at the beginning of Section 4.3, $F = \sum_i F_i$, where each of F_i generates an irreducible automorphic cuspidal representation π_i. Write $\pi_i = \otimes \pi_{i,p}$ and let $\Phi_i = \otimes \phi_{i,p}$ be the adelisation of F_i. Denote by F_i' the Siegelisation of F_i, $\Phi_i' = \otimes \phi_{i,p}'$ its adelisation, and let π_i' be the corresponding automorphic representation. From a representation-theoretic point of view, at each place p and for each i,

$$\phi_{i,p}' = \sum_{\gamma \in \Gamma^0(p^{n_p}\mathbf{Z}_p)\backslash \mathrm{SL}_2(\mathbf{Z}_p)} \pi_{i,p}(\begin{pmatrix} \gamma & \\ & {}^t\gamma^{-1} \end{pmatrix})\phi_{i,p}\,,$$

where $n_p = ord_p N_1$; so $\phi_{i,p}'$ is a linear combination of the elements in the vector space of $\pi_{i,p}$. Hence for each i, $\pi_{i,p}'$ is a subrepresentation of an irreducible representation π_i, and thus $\pi = \pi'$.

To prove the last part of the proposition, we first express Fourier coefficients of F' in terms of Fourier coefficients of F. By the definition,

$$F'(Z) = \sum_{\gamma \in \Gamma^0(N_1)\backslash \mathrm{SL}_2(\mathbf{Z})} F(\gamma Z\, {}^t\gamma) = \sum_T \sum_\gamma a(F, {}^t\gamma^{-1} T \gamma^{-1}) e(\mathrm{tr}\,(TZ)),$$

and thus

$$a(F',T) = \sum_{\gamma \in \mathrm{SL}_2(\mathbf{Z})/\Gamma^0(N_1)} a(F, {}^t\gamma T \gamma).$$

Therefore, if $F' \neq 0$, then $a(F, {}^t\gamma T\gamma) \neq 0$ for some $\gamma \in \mathrm{SL}_2(\mathbf{Z})/\Gamma^0(N_1)$ and $T \in \mathcal{P}_2$. Moreover, because F' is also an eigenform for the local Hecke algebra at primes $p \nmid N_2$, by Corollary 4.4.1 we can guarantee that $\operatorname{cont} T | N_2^\infty$. Hence, because the content and discriminant of ${}^t\gamma T \gamma$ are the same as for T, F has a non-zero Fourier coefficient $a(F, T_0)$ as stated in the hypothesis. □

Remark. An important assumption in the above proposition is that the Siegelisation of a non-zero modular form is non-zero. However, thanks to the second part of this proposition, we know that the associated automorphic representation stays the same. When N_1, N_2 are square-free, a necessary condition for the Siegelisation of a paramodular form to be non-zero is that the dimension of the space of P_{02}-fixed vectors is smaller or equal to the dimension of the space of P_1-fixed vectors for a fixed representation π_p at each place $p|N_1$. A quick look at Table 1 tells us that this is *not* the case only if π_p is of type VIc for some $p|N_1$.

4.5 Application to Maass relations

Classical Saito-Kurokawa lifts are the lifts from classical cusp forms $f \in S^{(1)}_{2k-2}(\mathrm{SL}_2(\mathbf{Z}))$ with k even to Siegel cusp forms $F \in S^{(2)}_k(\mathrm{Sp}_4(\mathbf{Z}))$. It is known (eg. [35]) that the space of these lifts consists precisely of the cusp forms whose coefficients satisfy the so-called *Maass relations*:

$$a(F, \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}) = \sum_{r|\gcd(a,b,c)} r^{k-1} a(F, \begin{pmatrix} \frac{ac}{r^2} & \frac{b}{2r} \\ \frac{b}{2r} & 1 \end{pmatrix}) . \tag{4.14}$$

However, the notion of a classical Saito-Kurokawa lift may be generalised to a lift from a cusp form $f \in S^{(1)}_{2k-2}(\Gamma_0(N))$ to $F \in S^{(2)}_k(\Gamma)$ for some congruence subgroup Γ. In fact, depending on a choice of generalisation, one may obtain more than one lift from a single $f \in S^{(1)}_{2k-2}(\Gamma_0(N))$ (cf. [54]). In any case, the automorphic representations containing the corresponding vectors are nearly equivalent, i.e. the local components are equivalent at almost every place; they are nearly equivalent to a constituent of a global induced representation of a proper parabolic subgroup of $G(\mathbf{A})$, and are called CAP (cuspidal associated to parabolic) representations.

The natural question arises: do the coefficients of the generalised lifts also satisfy a version of Maass relations? We are going to show that the answer is YES.

Let π, Φ, F be as in Section 4.3, that is F is a cuspidal Siegel modular form invariant under the action of $\Gamma_0(N_1, N_2)$ that is an eigenform of the local Hecke algebra at primes $p \nmid N_2$ and that gives rise to an irreducible automorphic representation π. Suppose that for primes $p \nmid N_2$, $\pi_p = \chi_p 1_{\mathrm{GL}(2)} \rtimes \chi_p^{-1}$ with an unramified character χ_p of $\mathbf{Q}_p^\times$ (a representation of type IIb according to Table 1). Note that these are non-tempered, non-generic representations. The set of π obtained in this way is precisely the set of CAP representations attached to the Siegel parabolic subgroup of $G(\mathbf{A})$ (cf. [9]).

Lemma 4.5.1. *For representation π as above, any vector $\tilde{\Phi} = \otimes_p \tilde{\phi}_p$ in the vector space V_π of π and any non-degenerate matrix $S \in M_2^{\mathrm{sym}}(\mathbf{Q})$, we have:*

$$\tilde{\Phi}_S(tg) = \tilde{\Phi}_S(g) \quad \textit{for all} \quad g \in G(\mathbf{A}) \quad \textit{and} \quad t \in \prod_{p \nmid N_2} T_S(\mathbf{Q}_p) \prod_{p | N_2} 1_2 \,.$$

Proof. Let $\tilde{\Phi} = \otimes_p \tilde{\phi}_p$ be as in the lemma, and let $\mathcal{S} = \{p : p|N_2\}$. Without loss of generality we may assume $g = 1_2$. Let $V^{\mathcal{S}}$ be the subspace of V_π generated by all vectors of the form $\otimes_{p \in \mathcal{S}} \tilde{\phi}_p \otimes_{p \notin \mathcal{S}} \psi_p$ with $\psi_p \in V_{\pi_p}$. The right action of $\otimes_{p \notin \mathcal{S}} G(\mathbf{Q}_p)$ on $V^{\mathcal{S}}$ makes $V^{\mathcal{S}}$ a representation isomorphic to $\otimes_{p \notin \mathcal{S}} \pi_p$. Define

$$\beta \colon V^{\mathcal{S}} \to \mathbf{C} \,, \qquad \beta(\Psi) := \Psi_S(1) = \int_{U(\mathbf{Q}) \backslash U(\mathbf{A})} \Psi(u) \theta_S^{-1}(u) du \,.$$

Note that $\beta(\pi(t)\Psi) = \Psi_S(t)$. We need to show that $\beta(\pi(t)\tilde{\Phi}) = \beta(\tilde{\Phi})$ for all $t \in \prod_{p \notin \mathcal{S}} T(\mathbf{Q}_p)$. This is trivial if $\beta \equiv 0$. So assume $\beta \not\equiv 0$. Let

$$\Phi' = \otimes_{p \in \mathcal{S}} \tilde{\phi}_p \otimes_{p \notin \mathcal{S}} \phi'_p \quad \text{be such that} \quad \beta(\Phi') \neq 0 \,.$$

For each $p \notin \mathcal{S}$ we get a functional β_p on V_{π_p} via

$$\beta_p(\psi_p) := \beta(\psi_p \otimes_{q \in \mathcal{S}} \tilde{\phi}_q \otimes_{q \notin \mathcal{S} \cup \{p\}} \phi'_q) \,.$$

Then $\beta_p(\phi'_p) \neq 0$ and thus β_p is a non-zero functional on V_{π_p}. Clearly, β_p satisfies

$$\beta_p(\pi_p(u)\psi_p) = \theta_S(u)\beta_p(\psi_p) \qquad \text{for all} \quad \psi_p \in V_{\pi_p} \text{ and } u \in U(\mathbf{Q}_p)$$

By Corollary 4.2 of [37], the matrix S satisfies the conditions of Lemma 4.1, [35], and therefore by this lemma

- the space of such functionals β_p is one-dimensional,
- $\beta_p(\pi_p(t)\psi_p) = \beta_p(\psi_p)$ for all $\psi_p \in V_{\pi_p}$ and $t \in T(\mathbf{Q}_p)$.

So there exists a constant $C_{\mathcal{S}}$ such that

$$\beta(\Psi) = C_{\mathcal{S}} \prod_{p \notin \mathcal{S}} \beta_p(\psi_p)$$

whenever $\Psi \in V^{\mathcal{S}}$ corresponds to $\otimes_{p\in\mathcal{S}}\tilde{\phi}_p \otimes_{p\notin\mathcal{S}} \psi_p$. Hence $\beta(\pi(t)\tilde{\Phi}) = \beta(\tilde{\Phi})$ for all $t \in \prod_{p\notin\mathcal{S}} T(\mathbf{Q}_p)$.

$\square$

Lemma 4.5.2. *Let F, N_1 be as above, $S = S(d)$, and L, M any positive integers. Then for any $c_1, c_2 \in \mathrm{Cl}_d(MN_1)$,*

$$a(F, L \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix}\right) S_{c_1} \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix}\right)) = a(F, L \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix}\right) S_{c_2} \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix}\right)).$$

Proof. Let $\{t_c\}_c$ be a set of representatives of $\mathrm{Cl}_d(MN_1)$. We may choose t_c so that $t_{c,p} = 1_2$ for all $p|N_2$. Indeed, if $\tilde{t} \in T(\mathbf{Q})$ is such that $\tilde{t}_p = t_{c,p}$ for all $p|N_2$, then $t_c = \tilde{t}\prod_{p|N_2} 1_2 \prod_{p\nmid N_2} \tilde{t}_p^{-1} t_{c,p}$. From the proofs of Theorem 4.2.1 and 4.3.1, and using their notation, we get

$$a(F, L \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix}\right) S_c \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix}\right)) = (LM)^k e^{2\pi \mathrm{tr}\, S} \frac{1}{r} \int_{\mathbf{R}^\times \backslash T(\mathbf{R})} \Phi_{\phi_{L,M}(c)}(H_\infty^{-1} m_c t_\infty) dt_\infty\,,$$

where

$$\Phi_{\phi_{L,M}(c)}(H_\infty^{-1} m_c t_\infty) = \Phi_S(t_c t_\infty H(L,M)) \overset{\text{Lemma 4.5.1}}{=} \Phi_S(t_\infty H(L,M))$$

does not depend on c. $\square$

Hence, it makes sense to write $a(F; dM^2, L)$ for any Fourier coefficient of F that is of the form $a(F, L \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix}\right) S_c \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix}\right))$ for some $c \in \mathrm{Cl}_d(MN_1)$, or in another words, for $a(F, T)$ with $T \in H_1(dM^2, L; \Gamma^0(N_1))$.

The following theorem generalises Theorem 5.1, [35] to cuspidal Siegel modular forms of level $\Gamma_0(N_1, N_2)$ with $N_2 > 1$.

Theorem 4.5.1. *Let F be as above and $\mathcal{S} = \{p : p|N_2\}$. For any fundamental discriminant d and any positive integers L, M, Fourier coefficients* *of F satisfy the following Maass relations:*

$$a(F; dM^2, L) = \sum_{\substack{r|L \\ \gcd(r,N_2)=1}} r^{k-1} a(F; d\left(\frac{LM}{rL_{\mathcal{S}}}\right)^2, L_{\mathcal{S}})\,. \tag{4.15}$$

Hence, if

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} = L\begin{pmatrix} M & \\ & 1 \end{pmatrix} S_c \begin{pmatrix} M & \\ & 1 \end{pmatrix}$$

for some $c \in \mathrm{Cl}_d(MN_1)$,

$$a(F, \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}) = \sum_{\substack{r|\gcd(a,b,c) \\ \gcd(r,N_2)=1}} r^{k-1} a(F, L_{\mathcal{S}} \begin{pmatrix} \frac{ac}{(rL_{\mathcal{S}})^2} & \frac{b}{2rL_{\mathcal{S}}} \\ \frac{b}{2rL_{\mathcal{S}}} & 1 \end{pmatrix})\,,$$

where $L_{\mathcal{S}} = (\gcd(a,b,c), N_2^\infty)$.

Proof. Recall Corollary 4.3.1. If we take Λ to be a trivial character[6] in the formula (4.11), and $M' = M_{\mathcal{S}}$, $L' = L_{\mathcal{S}}$,

$$a(F; dM^2, L) = \left(\frac{LM}{L_{\mathcal{S}}M_{\mathcal{S}}}\right)^k a(F; dM_{\mathcal{S}}^2, L_{\mathcal{S}}) \prod_{\substack{p|LM \\ p\notin\mathcal{S}}} B_{\phi_p}(h_p(l_p, m_p))\,,$$

where $l_p = \mathrm{o}rd_pL$, $m_p = \mathrm{o}rd_pM$. Similarly, for any divisor r of L,

$$a(F; d\left(\frac{LM}{rL_{\mathcal{S}}}\right)^2, L_{\mathcal{S}}) = \left(\frac{LM}{rL_{\mathcal{S}}M_{\mathcal{S}}}\right)^k a(F; dM_{\mathcal{S}}^2, L_{\mathcal{S}}) \prod_{\substack{p|LM/r \\ p\notin\mathcal{S}}} B_{\phi_p}(h_p(0, l_p+m_p-r_p))\,.$$

Note that the last product can actually be taken over primes $p|LM$ that are not in $\mathcal{S}$. Indeed, in the product over $p|LM/r, p \notin \mathcal{S}$ we miss only those elements for which $r_p = l_p$ and $m_p = 0$. But in this case $B_{\phi_p}(h_p(0, l_p+m_p-r_p)) = B_{\phi_p}(h_p(0,0)) = 1$ by Theorem 2.2.1.

Moreover, it is known ([35], Theorem 2.1) that the spherical vectors of the representations of type IIb satisfy the equation

$$B_p(h(l,m)) = \sum_{i=0}^{l} p^{-i} B_p(h(0, l+m-i))$$

[6] It makes sense to take $\Lambda = 1$, because the local representations that we consider here have a unique local Bessel model (cf. Table 2), and for this Bessel model $\Lambda_p = 1$.

for all $l, m \geq 0$. Hence, the equation (4.15) holds if and only if

$$\prod_{\substack{p|LM \\ p\notin\mathcal{S}}} \sum_{i=0}^{l} p^{-i} B_{\phi_p}(h_p(0, l_p + m_p - i)) = \sum_{r|L/L_{\mathcal{S}}} \frac{1}{r} \prod_{\substack{p|LM \\ p\notin\mathcal{S}}} B_{\phi_p}(h_p(0, l_p + m_p - r_p)) \,,$$

which is true. □

Corollary 4.5.1. *Let F, N_1, N_2 be as above. For any matrix $T = L\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ such that $\left(\frac{b^2-4ac}{p}\right) = -1$ for every $p|N_1$, and $L|N_2^\infty$,*

$$a(F, L\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}) = \sum_{\substack{r|\gcd(a,b,c) \\ \gcd(r,N_2)=1}} r^{k-1} a(F, L\begin{pmatrix} \frac{ac}{r^2} & \frac{b}{2r} \\ \frac{b}{2r} & 1 \end{pmatrix}).$$

Remark. One of the main differences between the classical Maass relations (4.14) and the Maass relations (4.15) is that the coefficients $a(F,T)$ on the right hand side of the first equality have the matrix T of content 1, and in particular, the $(2,2)$-entry of T equals 1. In general, Saito-Kurokawa lifts do not enjoy this property. Indeed, there exist paramodular forms that are Saito-Kurokawa lifts ([54]), but all their Fourier coefficients $a(F,T)$ have the $(2,2)$-entry of T divisible by N (cf. Section 5.1.1). Even though our result does not apply to paramodular forms (as representations of type IIb are not generic), we treat this fact as an indication that a further study of Fourier coefficients of Siegel modular forms of higher levels might be necessary to distinguish the cases where our result could be improved.

Chapter 5

Paramodular forms

In this chapter we present a result concerning non-vanishing of fundamental Fourier coefficients of certain paramodular forms. This result, together with the consecutive steps of its proof and limitations of classical methods, encouraged us to look from a broader point of view and inspired works on relations between Fourier coefficients, which we carried out in Chapter 4.

However, it turns out that the main relation (4.10) for Siegel modular forms that we obtained in Chapter 4 do not capture *any* primitive coefficient of paramodular forms. It is because, as we explain below, the matrices occurring in the Fourier expansion of paramodular forms always have the $(2,2)$-entry divisible by the level, whereas the relation (4.10) concerns only those whose $(2,2)$-entry is coprime to the level (cf. Section 3.3). This situation makes our main theorem of this chapter even more important.

The main objects of this chapter are paramodular forms and their Fourier coefficients. Nevertheless, especially in the first two sections, we refer to the objects that may be defined for other Siegel modular forms as well. We also provide historical background and motivation for our research.

5.1 Preliminaries

5.1.1 Paramodular forms and Jacobi forms

A holomorphic function $F\colon \mathcal{H}_2 \to \mathbf{C}$ defined on the Siegel upper half-space

$$\mathcal{H}_2 = \{X + iY : X, Y \in M_2(\mathbf{R}) \text{ symmetric}, Y \text{ positive definite}\}$$

is a *paramodular form* of weight k and level N if

$$F|_k\gamma(Z) = F(Z) \quad \text{for any} \quad \gamma \in \Gamma^{\mathrm{para}}(N)$$

according to the action (4.8), where

$$\Gamma^{\mathrm{para}}(N) := \mathrm{Sp}_4(\mathbf{Q}) \cap \begin{pmatrix} \mathbf{Z} & N\mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \mathbf{Z}/N \\ \mathbf{Z} & N\mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ N\mathbf{Z} & N\mathbf{Z} & N\mathbf{Z} & \mathbf{Z} \end{pmatrix}.$$

We are only interested in the case when F is a cusp form, that is, F vanishes at all the cusps of the group $\Gamma^{\mathrm{para}}(N)$; we denote this set by $S_k^{(2)}(\Gamma^{\mathrm{para}}(N))$. All Siegel modular forms that we encountered in Chapter 4, i.e. that are associated to irreducible automorphic cuspidal representations of $G(\mathbf{A})$, are cusp forms. As we saw, they have a unique Fourier expansion

$$F(Z) = \sum_{\substack{T={}^tT, T>0 \\ \text{half-integral}}} a(F,T)e(\mathrm{tr}\,(TZ))\,. \tag{5.1}$$

Moreover, it is easy to see that the Fourier coefficients $a(F,T)$ satisfy

$$a(F, {}^tATA) = a(F,T) \quad \text{for all } A \in \Gamma^0(N)\,. \tag{5.2}$$

If we expand (5.1) in terms of $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ and $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$, we obtain a Fourier-Jacobi expansion of F,

$$F(Z) = \sum_{\substack{m\geq 0 \\ 4nm-r^2\geq 0}} a(F, \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right))e(n\tau)e(rz)e(m\tau') = \sum_{m\geq 0} \phi_m(\tau,z)e(m\tau')\,,$$

where ϕ_m is a Jacobi form of weight k and index m. Jacobi forms of weight k, index m and level $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$, denoted $J_{k,m}(\Gamma)$ (or $J_{k,m}(N)$ if $\Gamma = \Gamma_0(N)$), are invariant under the action of $\Gamma \ltimes \mathbf{Z}^2$ as described in [12]. Actually, the group that acts on the space of Jacobi forms is a Jacobi group $\mathrm{GL}_2(\mathbf{R}) \ltimes \mathrm{H}_{\mathbf{R}}$, where $\mathrm{H}_{\mathbf{R}}$ is the Heisenberg group which is $\mathbf{R}^2 \times \mathbf{R}$ as a set.

Remark. The Jacobi group embeds into $\mathrm{GSp}_4(\mathbf{R})$ via

$$(g, ((\lambda,\mu),\kappa)) \longmapsto \left(\begin{smallmatrix} a & & b & \\ & \det g & & \\ c & & d & \\ & & & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & & & \mu \\ \lambda & 1 & \mu & \kappa \\ & & 1 & -\lambda \\ & & & 1 \end{smallmatrix}\right), \text{ where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The image of this map acts on the space $M_k^{(2)}(\Gamma^{(2)})$ for any congruence subgroup $\Gamma^{(2)}$ of $\mathrm{Sp}_4(\mathbf{Z})$, and agrees with the action of the Jacobi group. Hence, it is easy to see that if $\Gamma = \Gamma^{\mathrm{para}}(N)$, then each Fourier-Jacobi coefficient ϕ_m is a Jacobi form of level 1. Moreover, one can also show that for paramodular forms of level N, $\phi_m \neq 0$ only if $N|m$. In other words,

$$a(F, \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)) \neq 0\,, \quad F \in S_k^{(2)}(\Gamma^{\mathrm{para}}(N)) \qquad \Longrightarrow \qquad N|m\,.$$

This follows from the definition of F and comparing the coefficients in the equality

$$F(Z) = F\left(Z + \left(\begin{smallmatrix} 0 & \\ & 1/N \end{smallmatrix}\right)\right) = \sum_{\substack{m \geq 0 \\ 4nm - r^2 \geq 0}} a(F, \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)) e(n\tau) e(rz) e(m\tau') e(m/N).$$

In consequence, the relation (4.3.1) misses all the primitive coefficients of paramodular forms. It is because the set $H_1(dM^2, 1; \Gamma^0(N))$ that would occur in the summand for paramodular forms of level N, consists only of the elements that have the $(2,2)$-entry coprime to N, as we explained in Lemma 3.3.1.

5.1.2 Modular forms of half-integral weight

Modular forms of half-integral weight are holomorphic functions defined on the complex upper half plane $\mathcal{H}_1$ that are holomorphic at cusps and are invariant under the action of congruence subgroups of $\Gamma_0(4)$ in a sense similar to (4.8), but with $k \in \frac{1}{2}\mathbf{Z}$. However, if we left it without any change, the space of such functions would be zero for *any* congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$; see for example [23]. Hence we choose the square root having argument in $(-\pi/2, \pi/2]$ and change the automorphy factor $(cz+d)^k$ to the k-th power of $(-1)^{(d-1)/2}\left(\frac{c}{d}\right)\sqrt{cz+d}$ in case $c \neq 0$.[1] We denote the space of modular forms of weight k and level $\Gamma_0(4N)$ by $M_k^{(1)}(4N)$, and the subset of cusp forms by $S_k^{(1)}(4N)$.

We recall now a few useful facts. Let $\phi_m(\tau, z) \in J_{k,m}(\Gamma)$ be a Jacobi form coming from a Fourier-Jacobi expansion as above. It can also be written as

$$\phi_m(\tau, z) = \sum_{0 \leq \mu < 2m} h_\mu(\tau) \sum_{\substack{r \in \mathbf{Z} \\ r \equiv \mu \pmod{2m}}} e\left(\frac{r^2}{4m}\tau\right) e(rz)\,,$$

[1]N.B., $(-1)^{(d-1)/2}\left(\frac{c}{d}\right)\sqrt{cz+d} = \theta(\gamma z)/\theta(z)$ for $\gamma \in \Gamma_0(4)$.

where

$$h_\mu(\tau) = \sum_{\substack{D\geq 0 \\ D\equiv -\mu^2 \pmod{4m}}} a(F, \begin{pmatrix} \frac{D+\mu^2}{4m} & \mu/2 \\ \mu/2 & m \end{pmatrix}) e\left(\frac{D}{4m}\tau\right).$$

Note that the matrix $\begin{pmatrix} \frac{D+\mu^2}{4m} & \mu/2 \\ \mu/2 & m \end{pmatrix}$ has a discriminant $-D$.

Theorem 5.1.1 (Eichler, Zagier; [12]). *Consider a map that to a Jacobi form $\phi_m(\tau, z) \in J_{k,m}(1)$ attaches a function $h(\tau) := \sum_{0\leq\mu<2m} h_\mu(4m\tau)$. Then $h \in M^{(1)}_{k-1/2}(4m)$. Moreover, if m is prime and k even, such map is an isomorphism onto the space $M^{(1),+}_{k-1/2}(4m)$ of modular forms in $M^{(1)}_{k-1/2}(4m)$ whose D-th Fourier coefficient is zero for all D with $\left(\frac{-D}{m}\right) = -1$.*

This theorem was extended by Manickam and Ramakrishnan in [29] to Jacobi cusp forms of level $\Gamma_0(N)$ and even weight. They constructed a linear map $J^{\text{cusp}}_{k,m}(N) \to S^{(1)}_{k-1/2}(4mN)$ that commutes with the action of Hecke operators.

Theorem 5.1.1 gives us a tool to construct modular forms of half-integral weight out of Fourier-Jacobi expansion of paramodular forms; and, according to the remark above, we can use for this purpose *any* Jacobi form occurring in the expansion. The next theorem gives us an insight into the nature of Fourier coefficients of modular forms of half-integral weight. Because the coefficients $a(h_\mu, n)$ of the h_μ constructed above are defined in terms of the Fourier coefficients of a Siegel modular form, two theorems below will be crucial in our investigations of the Fourier coefficients of paramodular forms. The second one is especially important as it will allow us to reach the paramodular forms that occur in the statement of the paramodular conjecture.

Theorem 5.1.2 (Saha; [49]). *Let $k \geq 2$ and let N be a square-free integer. Suppose that $f \in S^{(1)}_{k+1/2}(4N)$ is non-zero. Then, one has the lower bound*

$$\#\{0 < D < X : D \text{ square-free}, a(f, D) \neq 0\} \gg_{f,\delta} X^\delta,$$

where $\delta > 0$ is an absolute constant (any value of $\delta < 5/8$ is admissible). In particular, there are infinitely many square-free integers D such that $a(f, D) \neq 0$.

Theorem 5.1.3 (Li; [27]). *Let N be an odd and square-free integer and $0 \neq f \in S^{(1)}_{3/2}(4N)$. Then for any finite set of primes $\mathcal{S}$, there are infinitely many square-free integers D satisfying $a(f, D) \neq 0$ and $\gcd(D, l) = 1$ for all $l \in \mathcal{S}$.*

Remark. If a modular form f is twisted by a character, one can slightly weaken the assumption about N in two theorems above (cf. [51], [27]). In any case, N cannot be divisible by cubes.

5.1.3 Hecke operators

As in the theory of classical modular forms, one can define Hecke operators on the space of Siegel modular forms of degree 2 ([1]). The ones of special interest to us are[2]

$$T(p) := \Gamma^{\mathrm{para}}(N) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix} \Gamma^{\mathrm{para}}(N)$$

and

$$T(p^2) := \Gamma^{\mathrm{para}}(N) \begin{pmatrix} 1 & & & \\ & p & & \\ & & p^2 & \\ & & & p \end{pmatrix} \Gamma^{\mathrm{para}}(N)$$

for $p \nmid N$, and

$$U(p) := \Gamma^{\mathrm{para}}(N) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix} \Gamma^{\mathrm{para}}(N)$$

for $p \mid N$. They act on the space of Siegel modular forms of degree 2 according to the following rule. If $\Gamma^{\mathrm{para}}(N)\alpha\Gamma^{\mathrm{para}}(N) = \coprod_i \Gamma^{\mathrm{para}}(N)\alpha_i$ is a coset decomposition, then

$$F|_k \Gamma^{\mathrm{para}}(N)\alpha\Gamma^{\mathrm{para}}(N) = F|_k \coprod_i \Gamma^{\mathrm{para}}(N)\alpha_i = \sum_i F_{|_k}\alpha_i \, ,$$

where the action $F|_k\alpha_i$ is as in (4.8). We will write down the action of the operators $U(p)$ and $T(p) + T(p^2)$ explicitly in Section 4.4.

We say that $F \in \Gamma^{\mathrm{para}}(N)$ is a *Hecke eigenform* if it is an eigenform of the operators $T(p), T(p^2)$ for all $p \nmid N$ and $U(p)$ for all $p|N$.

Another useful operator is the Fricke involution

$$\mu_N := \frac{1}{\sqrt{N}} \begin{pmatrix} & N & & \\ -1 & & & \\ & & & 1 \\ & & -N & \end{pmatrix} .$$

It normalizes $\Gamma^{\mathrm{para}}(N)$, and since $\mu_N^2 = -1_4$, the space $S_k^{(2)}(\Gamma^{\mathrm{para}}(N))$ decomposes into μ_N-eigenspaces $S_k^{(2)}(\Gamma^{\mathrm{para}}(N))^{\pm}$ with eigenvalues ± 1. If $F_{|_k}\mu_N = \epsilon F$ for $F \in S_k^{(2)}(\Gamma^{\mathrm{para}}(N))$, then the Fourier coefficients of F possess the symmetry

$$a(F, \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}) = \epsilon a(F, \begin{pmatrix} m/N & -r/2 \\ -r/2 & nN \end{pmatrix}) . \tag{5.3}$$

[2]Note the resmblence of these operators to the ones introduced in Theorem 2.3.3.

5.1.4 Newforms and oldforms

There are a few attempts to define a newform theory for Siegel modular forms of degree n greater than 1. In the classical case ($n = 1$), Atkin and Lehner [3] described the space of newforms as the orthogonal complement[3] of the space of oldforms, which consists of modular forms that are obtained from modular forms of lower level by the action of the $U(p)$ operators for $p|N$. Already when $n = 2$ the situation is more complicated as these two sets are completely different.

Ibukiyama and Katsurada [18] followed the classical Atkin-Lehner theory and for $\Gamma_0^{(2)}(N)$ defined a space of newforms as the orthogonal complement of

$$\mathrm{span}\{F(dZ) : F(Z) \in S_k^{(2)}(\Gamma_0^{(2)}(M)), dM|N, M \neq N\} \subset S_k^{(2)}(\Gamma_0^{(2)}(N)) \,, \tag{5.4}$$

so that the space (5.4) is the image of the $U(d)$ operators on $S_k^{(2)}(\Gamma_0^{(2)}(M))$ with $dM|N$. However, it is not clear why this space should be favoured as *the* space of newforms.

Roberts and Schmidt [45] gave a local picture on this subject in case of $\Gamma^{\mathrm{para}}(N)$. Instead of looking at the description of the classical newforms by the Fourier coefficients, they focused on their characterisation as certain vectors in the space of representations of $\mathrm{GL}_2(F)$ over a local field F. In this language the classical oldforms are obtained from the newforms by applying two level raising Hecke operators and taking linear combinations of those. In analogy to this characterisation where, in particular, the space of newforms of a fixed level is one-dimensional, they created a theory of the local newforms, in which the oldforms arise as an effect of applying three level raising Hecke operators and taking linear combinations. We should mention here that if a paramodular form is a newform in this sense, then it is an eigenfunction of the operators $T(p), T(p^2), U(p)$ and μ_N.

[3] The orthogonal complement is taken with respect to *Petersson inner product.* Given $F, G \in S_k^{(n)}(\Gamma)$ for some congruence subgroup Γ of $\mathrm{Sp}_{2n}(\mathbf{Z})$, their Petersson inner product is defined to be

$$(F, G) := \frac{1}{\mathrm{vol}(\Gamma\backslash\mathcal{H}_\mathrm{n})} \int_{\Gamma\backslash\mathcal{H}_n} F(Z)\overline{G(Z)} \det Y^{k-n-1} dX dY \,.$$

5.2 History and motivation

One of the most basic questions one might ask about Siegel modular forms is their determination by Fourier coefficients. As we mentioned in the introduction, we are not so much interested in the smallest subset of such coefficients, but rather in a set that might be also interesting from theoretical point of view.

Many results on this topic has resembled the fact that a classical cuspidal Hecke eigenform

$$f(z) = \sum_{n=1}^{\infty} a(f,n) e^{2\pi i n z} \in S_k^{(1)}(\mathrm{SL}_2(\mathbf{Z}))$$

is determined by the set

$$\{a(f,1)\} \cup \{a(f,p) : p \text{ prime}\} .$$

If

$$F(Z) = \sum_{T \in \mathcal{P}_n} a(F,T) e^{2\pi i \operatorname{tr}(TZ)} \in S_k^{(n)}(\mathrm{Sp}_{2n}(\mathbf{Z})) ,$$

the analogous condition on the $n \times n$ matrices T might be

$$\{a(F,T) : \operatorname{disc} T \text{ square-free}\} .$$

What conditions would we need to put on F so that this could be true?

There are several results that concern Hecke eigenforms of full level (cf. introduction to [62]). However, none of them goes beyond simplifying the content of T. Therefore it is worth to note a result due to Zagier [63], which states that *every* non-zero Siegel modular form F of level $\mathrm{Sp}_4(\mathbf{Z})$ is determined by its *primitive Fourier coefficients*, i.e. by $\{a(F,T) : \operatorname{cont} T = 1\}$. Many years later it was generalised by Yamana both to higher levels and higher degree:

Theorem 5.2.1 (Yamana; [62]). *Let F be a Siegel modular form of degree $n \geq 2$, weight $k \geq 1/2$ and level $\Gamma_0^{(n)}(N)$. Assume that $a(F,T) = 0$ for all $T \in \mathcal{P}_n$ such that $\operatorname{cont} T | N$. Then $F = 0$.*

This theorem was used by Ibukiyama and Katsurada [18] who under an additional assumption that F is in a newspace simplified the set determining F to $\{a(F,T) : \operatorname{cont} T = 1\}$. At the same time they showed that in order to be able to state such a result, the additional assumption was necessary.

Another breakthrough was a theorem of Saha [51] who showed, using Zagier's theorem, that non-zero cuspidal Siegel modular forms of level $\mathrm{Sp}_4(\mathbf{Z})$ are determined by their fundamental Fourier coefficients. Under additional assumptions and slight modifications of the proof, Saha and Schmidt obtained a couple of results for modular forms of higher levels.

Theorem 5.2.2 (Saha, Schmidt, [52]; Saha, [49])**.** *Let F be a non-zero cuspidal Siegel modular form of level $\Gamma_0^{(2)}(N)$ with N square-free, and even weight $k > 2$. Assume that one of the following conditions holds.*

1. *F is an eigenfunction for the $U(p)$ operator for all primes $p|N$.*
2. *F is in a new space.*

Then, for any $0 < \delta < 5/8$, one has the lower bound

$$|\{0 < D < X : D \text{ square-free, } \exists_T (a(F,T) \neq 0, \operatorname{disc} T = -D\}| \gg_{F,\delta} X^{\delta}.$$

In particular, F has infinitely many non-zero fundamental Fourier coefficients.

The proofs of both statements are very similar, and the difference in the assumptions comes from the fact that the first one uses Yamana's theorem, and the second one the theorem due to Ibukiyama and Katsurada. These assumptions assure the existence of a non-zero primitive Fourier coefficient, and thus non-vanishing of a Fourier-Jacobi coefficient of a prime index. This in turn allows to construct a non-zero modular form of half-integral weight that satisfies the conditions of Theorem 5.1.2 (Theorem 5.1.3 was not known then), and eventually implies non-vanishing of fundamental Fourier coefficients. Here very helpful are the theorems due to Eichler, Zagier and Manickam, Ramakrishnan mentioned in Section 5.1.2, which guarantee that the constructed function is indeed a modular form of half-integral weight of the prescribed level. The lack of these or non-square-free index of a Jacobi form make it very hard to use the key Theorem 5.1.2 for this method.

In the case of paramodular forms of level N it is always the case that the index of Fourier-Jacobi coefficients is divisible by N. Hence, unless a paramodular form is twisted by a suitable character and N is not divisible by cubes (cf. Theorem 5.1.1, 5.1.2 and the remark below them), it seems impossible to take this approach. On the other hand, there is no Yamana-like result in this situation, that is, there is no result that ensures the existence of $a(F,T) \neq 0$ with the content of T divisible by at most finitely many fixed prime numbers. Nevertheless, under an additional assumption that a paramodular form F of square-free level is an eigenform of certain Hecke operators, we prove in the next section that F has infinitely many non-zero fundamental Fourier coefficients.

5.3 Non-vanishing of fundamental Fourier coefficients

We state first our main theorem.

Theorem 5.3.1. *Let $0 \neq F \in S_k^{(2)}(\Gamma^{para}(N))$, with $k \geq 2$ even and N square-free, be an eigenfunction of the operators $T(p) + T(p^2)$ for primes $p \nmid N$, $U(p)$ for $p \mid N$ and μ_N. Then F has infinitely many nonzero fundamental Fourier coefficients.*

Remark. In particular, Theorem 5.3.1 holds for paramodular newforms.

The proof follows the strategy discussed at the end of the previous section. First we prove that F has a non-zero primitive Fourier coefficient. Then we construct a modular form of half-integral weight and use Theorems 5.1.2, 5.1.3.

Remark. As we noted at the end of Section 5.1.1 the information on existence of a non-zero primitive Fourier coefficient cannot be obtained from Theorem 4.3.1 as the coefficients $a(F, T)$ that occur there are supported on $\bigcup_{d,M,L} H_1(dM^2, L; \Gamma^0(N))$. In some cases one may Siegelise a paramodular form (cf. Definition 4.13 and propositions below) and be able to use e.g. Yamana's theorem. However, this relies on the assumption that Siegelisation gives a *non-zero* Siegel modular form, which is not always the case (cf. Remark at the end of Section 4.4).

Lemma 5.3.1. *Let $F \in S_k^{(2)}(\Gamma^{para}(N))$ be a nonzero paramodular form and $p||N$ be a prime. If F is an eigenform of the $U(p)$ operator with an eigenvalue λ, then the coefficients of F satisfy the following equality:*

$$\begin{aligned}\lambda a(F,T) &= p^{-k+3}a(F,pT) + p^k a\left(F, \frac{1}{p}T\right) \qquad (5.5)\\ &\quad - a\left(F, \frac{1}{p}\begin{pmatrix} \alpha p & 1 \\ -N\beta & p \end{pmatrix} T \begin{pmatrix} \alpha p & -N\beta \\ 1 & p \end{pmatrix}\right)\\ \textit{(if } p|m\textit{)} &\quad + p\sum_{b \in \mathbf{Z}/p\mathbf{Z}} a\left(F, \frac{1}{p}\begin{pmatrix} 1 & b \\ & p \end{pmatrix} T \begin{pmatrix} 1 & \\ b & p \end{pmatrix}\right)\end{aligned}$$

$$\begin{aligned}
&\textit{(if } p|n\textit{)} \qquad + (-1)^k p \sum_{b \in \mathbf{Z}/p\mathbf{Z}} a\left(F, \frac{1}{p}\begin{pmatrix} p & \\ -bN & -1 \end{pmatrix} T \begin{pmatrix} p & -bN \\ & -1 \end{pmatrix}\right) \\
&\textit{(if } p|r\textit{)} \qquad + pa\left(F, \frac{1}{p}\begin{pmatrix} \alpha p & 1 \\ -N\beta & p \end{pmatrix} T \begin{pmatrix} \alpha p & -N\beta \\ 1 & p \end{pmatrix}\right),
\end{aligned}$$

where $T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & mN \end{smallmatrix}\right)$, *and* $\alpha, \beta \in \mathbf{Z}$ *are such that* $\alpha p^2 + \beta N = p$. *(We take the convention* $a\left(F, \frac{1}{p} X\right) := 0$, *if* $p \nmid \operatorname{cont}(X)$*.)*

Proof. Lemma 6.1.2 of [45] gives coset representatives at the place p of the double coset defining the operator $U(p)$,[4]

$$\begin{aligned}
P_{02} \left(\begin{smallmatrix} I_2 & \\ & pI_2 \end{smallmatrix}\right) P_{02} = &\coprod_{a,b,c \in \mathbf{Z}/p\mathbf{Z}} P_{02} \left(\begin{smallmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & & a & b \\ & 1 & b & c/p \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right) \\
&\sqcup \coprod_{a,c \in \mathbf{Z}/p\mathbf{Z}} P_{02} \left(\begin{smallmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & p \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & & & \\ -a & 1 & & c/p \\ & & 1 & a \\ & & & 1 \end{smallmatrix}\right) \\
&\sqcup \coprod_{a,b \in \mathbf{Z}/p\mathbf{Z}} P_{02} \left(\begin{smallmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & & a & b \\ & 1 & b & \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & & & \\ & & & 1/p \\ & & 1 & \\ & -p & & \end{smallmatrix}\right) \\
&\sqcup \coprod_{a \in \mathbf{Z}/p\mathbf{Z}} P_{02} \left(\begin{smallmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & p \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & & & \\ -a & 1 & & \\ & & 1 & a \\ & & & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & & & \\ & & & 1/p \\ & & 1 & \\ & -p & & \end{smallmatrix}\right)
\end{aligned}$$

In fact, we can exchange a matrix $\left(\begin{smallmatrix} 1 & & & \\ & & & 1/p \\ & & 1 & \\ & -p & & \end{smallmatrix}\right)$ above by $\left(\begin{smallmatrix} 1 & & & \\ & & & 1/N \\ & & 1 & \\ & -N & & \end{smallmatrix}\right)$, and that will give us the same coset representatives. Moreover, at the place $q \neq p$, $P_{02} \left(\begin{smallmatrix} I_2 & \\ & pI_2 \end{smallmatrix}\right) P_{02} = P_{02}$, so using Chinese remainder theorem, we can choose:

$$\begin{aligned}
&\Gamma^{\mathrm{para}}(N) \left(\begin{smallmatrix} I_2 & \\ & pI_2 \end{smallmatrix}\right) \Gamma^{\mathrm{para}}(N) \\
&\quad = \coprod_{a,b,c \in \mathbf{Z}/p\mathbf{Z}} \Gamma^{\mathrm{para}}(N) \left(\begin{smallmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & & a & b \\ & 1 & b & c/p \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right) \\
&\quad \sqcup \coprod_{a,c \in \mathbf{Z}/p\mathbf{Z}} \Gamma^{\mathrm{para}}(N) \left(\begin{smallmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & p \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & & & \\ -a & 1 & & c/p \\ & & 1 & a \\ & & & 1 \end{smallmatrix}\right) \\
&\quad \sqcup \coprod_{a,b \in \mathbf{Z}/p\mathbf{Z}} \Gamma^{\mathrm{para}}(N) \left(\begin{smallmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & & a & b \\ & 1 & b & \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & & & \\ & & & 1/N \\ & & 1 & \\ & -N & & \end{smallmatrix}\right) \\
&\quad \sqcup \coprod_{a \in \mathbf{Z}/p\mathbf{Z}} \Gamma^{\mathrm{para}}(N) \left(\begin{smallmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & p \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & & & \\ -a & 1 & & \\ & & 1 & a \\ & & & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & & & \\ & & & 1/N \\ & & 1 & \\ & -N & & \end{smallmatrix}\right).
\end{aligned}$$

Using the invariance of F under the action of the paramodular group

[4] The coset representatives obtained in [45] are adjusted to our (classical) definition of P_{02}.

$\Gamma^{\mathrm{para}}(N)$, the coset representatives of $\Gamma^{\mathrm{para}}(N)\left(\begin{smallmatrix} I_2 & \\ & pI_2 \end{smallmatrix}\right)\Gamma^{\mathrm{para}}(N)$ act on F in the following way (unless stated otherwise, a matrix T occurring in the summand is of the form $\left(\begin{smallmatrix} n & r/2 \\ r/2 & mN \end{smallmatrix}\right)$):

$$
\begin{aligned}
F|_k & \coprod_{a,b,c\in\mathbf{Z}/p\mathbf{Z}} \Gamma^{\mathrm{para}}(N)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix}\begin{pmatrix} 1 & & a & b \\ & 1 & b & c/p \\ & & 1 & \\ & & & 1 \end{pmatrix}(Z) \\
&= p^{-k}\sum_{a,b,c\in\mathbf{Z}/p\mathbf{Z}} F\left(\frac{1}{p}Z+\frac{1}{p}\left(\begin{smallmatrix} a & b \\ b & c/p \end{smallmatrix}\right)\right) \\
&= p^{-k}\sum_{T} a(F,T)e\left(\mathrm{tr}\left(\frac{1}{p}TZ\right)\right)\sum_{a,b,c\in\mathbf{Z}/p\mathbf{Z}} e\left(\frac{na}{p}\right)e\left(\frac{rb}{p}\right)e\left(\frac{mNc}{p^2}\right) \\
&= p^{-k+3}\sum_{\substack{T \\ p|n,m,r}} a(F,T)e\left(\mathrm{tr}\left(\frac{1}{p}TZ\right)\right) \\
&= p^{-k+3}\sum_{T} a(F,pT)e(\mathrm{tr}\,(TZ)), \\
F|_k & \coprod_{a,c\in\mathbf{Z}/p\mathbf{Z}} \Gamma^{\mathrm{para}}(N)\begin{pmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & p \end{pmatrix}\begin{pmatrix} 1 & & & \\ -a & 1 & & c/p \\ & & 1 & a \\ & & & 1 \end{pmatrix}(Z) \\
&= \sum_{a,c\in\mathbf{Z}/p\mathbf{Z}} F\left(\left(\begin{pmatrix} p & \\ -a & 1 \end{pmatrix}Z+\left(\begin{smallmatrix} 0 & \\ & c/p \end{smallmatrix}\right)\right)\frac{1}{p}\left(\begin{smallmatrix} p & -a \\ & 1 \end{smallmatrix}\right)\right) \\
&= \sum_{T} a(F,T)\sum_{a\in\mathbf{Z}/p\mathbf{Z}} e\left(\mathrm{tr}\left(\frac{1}{p}\left(\begin{smallmatrix} p & -a \\ & 1 \end{smallmatrix}\right)T\left(\begin{smallmatrix} p & \\ -a & 1 \end{smallmatrix}\right)Z\right)\right) \\
&\quad \cdot \sum_{c\in\mathbf{Z}/p\mathbf{Z}} e\left(\mathrm{tr}\left(\frac{1}{p}\left(\begin{smallmatrix} p & -a \\ & 1 \end{smallmatrix}\right)T\left(\begin{smallmatrix} 0 & \\ & c/p \end{smallmatrix}\right)\right)\right) \\
&= p\sum_{\substack{T \\ p|m}}\sum_{a\in\mathbf{Z}/p\mathbf{Z}} a\left(F,\frac{1}{p}\left(\begin{smallmatrix} 1 & a \\ & p \end{smallmatrix}\right)T\left(\begin{smallmatrix} 1 & \\ a & p \end{smallmatrix}\right)\right)e(\mathrm{tr}\,(TZ)), \\
F_{|_k} & \coprod_{a,b\in\mathbf{Z}/p\mathbf{Z}} \Gamma^{\mathrm{para}}(N)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix}\begin{pmatrix} 1 & & a & b \\ & 1 & b & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & & & 1/N \\ & & 1 & \\ & -N & & \end{pmatrix}(Z) \\
&= \sum_{a,b\in\mathbf{Z}/p\mathbf{Z}}\left(F_{|_k}\begin{pmatrix} 1 & & & \\ & & & 1/N \\ & & 1 & \\ & -N & & \end{pmatrix}\begin{pmatrix} 1 & -bN & a & \\ & & b & 1/N \\ & & p & \\ & -pN & & \end{pmatrix}\right)(Z) \\
&= \sum_{a,b\in\mathbf{Z}/p\mathbf{Z}}\left(F_{|_k}\begin{pmatrix} 1 & -bN & a & \\ & -p & & \\ & & p & \\ & & -bN & -1 \end{pmatrix}\right)(Z) \\
&= (-1)^k\sum_{a,b\in\mathbf{Z}/p\mathbf{Z}}\sum_{T} a(F,T)e\Big(\mathrm{tr}\left(\left(\begin{smallmatrix} p & \\ -bN & -1 \end{smallmatrix}\right)^{-1}T\left(\left(\begin{smallmatrix} 1 & -bN \\ & -p \end{smallmatrix}\right)Z+\left(\begin{smallmatrix} a & \\ & 0 \end{smallmatrix}\right)\right)\right)\Big) \\
&= (-1)^k\sum_{b\in\mathbf{Z}/p\mathbf{Z}}\sum_{T} a(F,T)e\Big(\mathrm{tr}\left(\left(\begin{smallmatrix} p & \\ -bN & -1 \end{smallmatrix}\right)^{-1}T\left(\begin{smallmatrix} 1 & -bN \\ & -p \end{smallmatrix}\right)Z\right)\Big)\sum_{a\in\mathbf{Z}/p\mathbf{Z}} e\Big(\frac{na}{p}\Big) \\
&= p(-1)^k\sum_{\substack{T \\ p|n}}\sum_{b\in\mathbf{Z}/p\mathbf{Z}} a\left(F,\frac{1}{p}\left(\begin{smallmatrix} p & \\ -bN & -1 \end{smallmatrix}\right)T\left(\begin{smallmatrix} p & -bN \\ & -1 \end{smallmatrix}\right)\right)e(\mathrm{tr}\,(TZ)),
\end{aligned}
$$

$$
\begin{aligned}
F|_k &\coprod_{a\in \mathbf{Z}/p\mathbf{Z}} \Gamma^{\mathrm{para}}(N) \begin{pmatrix} p &&& \\ & 1 && \\ && 1 & \\ &&& p \end{pmatrix} \begin{pmatrix} 1 &&& \\ -a & 1 && \\ && 1 & a \\ &&& 1 \end{pmatrix} \begin{pmatrix} 1 &&& \\ &&& 1/N \\ && 1 & \\ & -N && \end{pmatrix} (Z) \\
&= F|_k \begin{pmatrix} p &&& \\ &&& 1/N \\ && 1 & \\ & -pN && \end{pmatrix} (Z) + \sum_{a\in(\mathbf{Z}/p\mathbf{Z})^\times} F|_k \begin{pmatrix} p &&& \\ -a &&& 1/N \\ & -aN & 1 & \\ & -pN && \end{pmatrix} (Z)\,.
\end{aligned}
$$

Before we can proceed further, we should investigate the case $a \neq 0$. We want to construct a matrix $g \in \Gamma^{\mathrm{para}}(N)$ so that if we substitute $F|_k g$ in place of $F|_k$ and consider the action of the above coset representative, we will obtain a Siegel parabolic matrix[5]. Let $\bar{a} := a^{-1} \bmod p$ and $\alpha, \beta \in \mathbf{Z}$ such that $\alpha p^2 + \beta N = p$ (the existence of α, β follows from the assumption that $p^2 \nmid N$), and put

$$
g := \begin{pmatrix} 1 & & -\beta\bar{a} & \beta(a\bar{a}-1)/p \\ (a\bar{a}-1)/p & \bar{a} & & -\alpha/N \\ aN/p & N & \alpha p & -\alpha a \\ Na & Np & -N\beta & N\beta a/p \end{pmatrix}.
$$

One can easily check that $g \in \Gamma^{\mathrm{para}}(N)$. Now that

$$
g \begin{pmatrix} p &&& \\ -a &&& 1/N \\ & -aN & 1 & \\ & -pN && \end{pmatrix} = \begin{pmatrix} p & N\beta & -\beta\bar{a} & \\ -1 & \alpha p & & \bar{a}/N \\ && \alpha p & 1 \\ && -N\beta & p \end{pmatrix},
$$

we are ready to determine the action of the coset representatives of the last type on F. Namely, the terms above can be written as:

$$
\begin{aligned}
F|_k &\begin{pmatrix} 1 &&& \\ &&& -1/N \\ && 1 & \\ & N && \end{pmatrix} \begin{pmatrix} p &&& \\ &&& 1/N \\ && 1 & \\ & -pN && \end{pmatrix} (Z) + \sum_{a\in(\mathbf{Z}/p\mathbf{Z})^\times} F|_k g \begin{pmatrix} p &&& \\ -a &&& 1/N \\ & -aN & 1 & \\ & -pN && \end{pmatrix} (Z) \\
&= F|_k \begin{pmatrix} p &&& \\ & p && \\ && 1 & \\ &&& 1 \end{pmatrix} (Z) + \sum_{a\in(\mathbf{Z}/p\mathbf{Z})^\times} F|_k \begin{pmatrix} p & N\beta & -\beta\bar{a} & \\ -1 & \alpha p & & \bar{a}/N \\ && \alpha p & 1 \\ && -N\beta & p \end{pmatrix} (Z) \\
&= p^k F(pZ) + \sum_{a\in(\mathbf{Z}/p\mathbf{Z})^\times} F\left(\left(\left(\begin{smallmatrix} p & N\beta \\ -1 & \alpha p \end{smallmatrix}\right) Z + \begin{pmatrix} -\beta\bar{a} & \\ & \bar{a}/N \end{pmatrix}\right)\left(\begin{smallmatrix} \alpha p & 1 \\ -N\beta & p \end{smallmatrix}\right)^{-1}\right)(Z) \\
&= p^k F(pZ) + \sum_{a\in(\mathbf{Z}/p\mathbf{Z})^\times} \sum_T a(F,T) e\left(\operatorname{tr}\left(\left(\begin{smallmatrix} \alpha p & 1 \\ -N\beta & p \end{smallmatrix}\right)^{-1} T \left(\begin{smallmatrix} p & N\beta \\ -1 & \alpha p \end{smallmatrix}\right) Z\right)\right) \\
&\quad \cdot e\left(\operatorname{tr}\left(\frac{\bar{a}}{p}\left(\begin{smallmatrix} n & r/2 \\ r/2 & mN \end{smallmatrix}\right)\begin{pmatrix} -\beta & \\ & 1/N \end{pmatrix}\left(\begin{smallmatrix} p & -1 \\ N\beta & \alpha p \end{smallmatrix}\right)\right)\right)
\end{aligned}
$$

[5]One can easily check that such a matrix g does not exist if $p^2|N$.

$$
\begin{aligned}
&= p^k F(pZ) + \sum_T a(F,T) e\left(\mathrm{tr}\left(\left(\begin{smallmatrix} \alpha p & 1 \\ -N\beta & p \end{smallmatrix}\right)^{-1} T \left(\begin{smallmatrix} p & N\beta \\ -1 & \alpha p \end{smallmatrix}\right) Z\right)\right) \\
&\quad \cdot \sum_{a \in (\mathbf{Z}/p\mathbf{Z})^\times} e\left(\frac{\bar{a}\beta r}{p}\right) \\
&= p^k \sum_T a\left(F, \frac{1}{p} T\right) e(\mathrm{tr}\,(TZ)) \\
&\quad + \sum_T \sum_{a \in (\mathbf{Z}/p\mathbf{Z})^\times} e\left(\frac{a\beta r}{p}\right) a\left(F, \frac{1}{p} \left(\begin{smallmatrix} \alpha p & 1 \\ -N\beta & p \end{smallmatrix}\right) T \left(\begin{smallmatrix} \alpha p & -N\beta \\ 1 & p \end{smallmatrix}\right)\right) e(\mathrm{tr}\,(TZ)).
\end{aligned}
$$

Hence, because $F|_k U(p) = \lambda F$, we obtain the equality

$$
\begin{aligned}
\lambda a(F,T) &= p^{-k+3} a(F, pT) + p^k a\left(F, \frac{1}{p} T\right) \\
&\quad - a\left(F, \frac{1}{p} \begin{pmatrix} \alpha p & 1 \\ -N\beta & p \end{pmatrix} T \begin{pmatrix} \alpha p & -N\beta \\ 1 & p \end{pmatrix}\right) \\
\text{(if } p|m) &\quad + p \sum_{b \in \mathbf{Z}/p\mathbf{Z}} a\left(F, \frac{1}{p} \begin{pmatrix} 1 & b \\ & p \end{pmatrix} T \begin{pmatrix} 1 & \\ b & p \end{pmatrix}\right) \\
\text{(if } p|n) &\quad + (-1)^k p \sum_{b \in \mathbf{Z}/p\mathbf{Z}} a\left(F, \frac{1}{p} \begin{pmatrix} p & \\ -bN & -1 \end{pmatrix} T \begin{pmatrix} p & -bN \\ & -1 \end{pmatrix}\right) \\
\text{(if } p|r) &\quad + pa\left(F, \frac{1}{p} \begin{pmatrix} \alpha p & 1 \\ -N\beta & p \end{pmatrix} T \begin{pmatrix} \alpha p & -N\beta \\ 1 & p \end{pmatrix}\right),
\end{aligned}
$$

where $\alpha, \beta \in \mathbf{Z}$ are such that $\alpha p^2 + \beta N = p$. □

Thanks to Lemma 5.3.1 we will be able to prove that F has a non-zero coefficient $a(F,T)$ with $\gcd(\mathrm{cont}\, T, N) = 1$. To get a non-zero primitive Fourier coefficient, we need to investigate the action of Hecke operators at $p \nmid N$. It turns out that the following result due to Evdokimov will be enough[6].

Proposition 5.3.2 (Evdokimov; [13]). *Let $F \in S_k^{(2)}(\Gamma^{para}(N))$. Assume that $F|_k T(p) + T(p^2) = \lambda F$. Then, using the notation of [13], the Fourier coefficients of F satisfy the relation*

$$
\lambda a(F,T) = a(F, pT) + p^{2k-3} a\left(F, \frac{1}{p} T\right) \tag{5.6}
$$

[6]Evdokimov considered Siegel modular forms with respect to principal congruence subgroup, but the Hecke algebras coincide at primes not dividing N.

$$+ p^{k-2} \sum_{U \in R(N) \subseteq \Gamma_0(N)} a\left(F, \frac{1}{p} \left(\begin{smallmatrix} 1 & \\ & p \end{smallmatrix}\right) U T\, {}^t\!U \left(\begin{smallmatrix} 1 & \\ & p \end{smallmatrix}\right)\right).$$

Lemma 5.3.3. *Let $F \in S_k^{(2)}(\Gamma^{para}(N))$ be a nonzero paramodular form of square-free level N that is an eigenform of the operators $T(p) + T(p^2)$ and $U(p)$ for all primes p. Then there exists a primitive matrix S for which $a(F,S) \neq 0$.*

Proof. This follows from close observation of behaviour of Fourier coefficients under the action of operators $U(p)$ and $T(p)+T(p^2)$, relations (5.5) and (5.6). Let $\mathcal{A}$ be the set of matrices S such that $a(F,S) \neq 0$. Let S' be the matrix in $\mathcal{A}$ whose discriminant is smallest. We claim that S' is primitive. If not, say $p \mid \operatorname{cont} S$ and $S = pT$, then, using the relations (5.5) and (5.6), we can find another matrix $S'' \in \mathcal{A}$ whose discriminant is smaller than $\operatorname{disc} S'$. Indeed, note that every coefficient occurring in (5.5) and (5.6), except $a(F,pT)$, has a discriminant that divides $\operatorname{disc} T$. This leads to a contradiction. $\square$

Now, having established the existence of a primitive matrix S for which $a(F,S) \neq 0$, we can move to the second part of the proof of Theorem 5.3.1.

Lemma 5.3.4. *Let $F \in S_k^{(2)}(\Gamma^{para}(N))$ be an eigenfunction of the operator μ_N. Assume that there is a primitive matrix $S = \left(\begin{smallmatrix} n & r/2 \\ r/2 & Nm \end{smallmatrix}\right)$ such that $a(F,S) \neq 0$. Then there exists an odd prime p not dividing N for which $\phi_{Np} \neq 0$.*

Proof. We will use the properties (5.2) and (5.3) of Fourier coefficients listed above. Let

$$S' := \begin{pmatrix} m & -r/2 \\ -r/2 & Nn \end{pmatrix} \quad \text{and} \quad A := \begin{pmatrix} a & Nc \\ b & d \end{pmatrix} \in \Gamma^0(N).$$

Then

$$a(F, {}^tAS'A) = a(F,S') = \epsilon a(F,S) \neq 0$$

and the right bottom entry of $AS'\,{}^tA$ is $N(c^2Nm - cdr + d^2n)$. Because $\gcd(n,r,Nm) = 1$, the form $c^2Nm - cdr + d^2n$ represents infinitely many primes ([61]). Let $c,d \in \mathbf{Z}$ be such that we obtain an odd prime p not dividing N. Then $\gcd(cN,d) = 1$, so we can find a,b so that $A \in \mathrm{SL}_2(\mathbf{Z})$. Hence, $\phi_{Np} \neq 0$. $\square$

After all that preparation, the proof of Theorem 5.3.1 will be very short:

Proof. We know from Lemma 5.3.3 and 5.3.4 that there exists an odd prime $p \nmid N$ such that $\phi_{Np} \not\equiv 0$. Define

$$h(\tau) := \sum_{D=1}^{\infty} \sum_{\substack{0 \le r < 2Np \\ r^2 \equiv -D \pmod{4Np}}} a\left(F, \begin{pmatrix} \frac{D+r^2}{4Np} & r/2 \\ r/2 & Np \end{pmatrix}\right) e(D\tau) = \sum_{D=1}^{\infty} a(h, D) e(D\tau)\,.$$

By Theorem 5.1.1, $0 \neq h \in M_{k-1/2}(4Np)$. Hence, by Lemma 5.3.4 and Theorems 5.1.2, 5.1.3, there are infinitely many square-free D for which $a(h, D) \neq 0$. For each such D there exists r such that $a\left(F, \begin{pmatrix} \frac{D+r^2}{4Np} & r/2 \\ r/2 & Np \end{pmatrix}\right) \neq 0$. □

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